



Analyzing multipartite entanglement via Schmidt decomposition in pure non-Gaussian three-mode states

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ABSTRACT

We investigate the distribution of entanglement in a system of three interacting quantum particles modeled as coupled harmonic oscillators using the Schmidt decomposition formalism. This approach enables us to represent the multipartite quantum state as a sum of product states between subsystems. The Schmidt coefficients precisely measure quantum correlations. By analyzing how these coefficients evolve as inter-oscillator coupling strengths vary, we reveal how entanglement is shared and redistributed across different bipartitions of the system. Through detailed numerical simulations, we identify coupling regimes that enhance, suppress, or destabilize entanglement, revealing the intricate structure of multipartite quantum correlations. Our findings provide valuable insights into controlling, and optimizing entanglement in networked quantum architectures. They offer practical guidance for quantum information processing and communication protocols. In the appropriate limit, our analysis naturally recovers known results for two coupled oscillators, thus validating the generality of our approach.

1. Introduction

Entanglement is a uniquely quantum mechanical phenomenon with no classical analog that lies at the heart of many phenomena in physics. It has been extensively studied in the context of bipartite systems, i.e., systems divided into two parts [1–5]. Many investigations have been conducted on the entanglement properties of bipartite systems, including coupled harmonic oscillators [6–8], spin systems [9,10], and itinerant particle models [11,12]. Recent works have also analyzed nonclassical photon statistics and generation of bipartite entanglement in excited coherent states [13], in addition to single-mode Gaussian coherence [14]. For a pure state of two parties, A and B , the Schmidt decomposition [15] provides the most useful quantification and characterization of entanglement. This decomposition expresses the state as a unique superposition of product states involving the A and B subsystems. The corresponding Schmidt coefficients quantify the amount of bipartite entanglement. Makarov studied the solutions of coupled harmonic oscillators and their quantum entanglement [16]. As a result, the analytical solutions of the nonstationary Schrödinger equation were derived and the explicit expressions for the Schmidt modes in both stationary and dynamical regimes are obtained. Additionally, the quantum entanglement of the system is analyzed via Schmidt modes. It is found that, for certain parameter regimes, the entanglement can become extremely large.

Multipartite entanglement, involving more than two parties, has a much richer structure than bipartite entanglement. It plays a central role in quantum information tasks, such as quantum secret sharing [17,18]. Its complexity stems from the large number of

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possible partitions and the variety of quantum correlations that can occur. For three parties, A , B , and C , in a pure state, a natural generalization of the Schmidt decomposition allows the global state to be expressed as a superposition of product states across different bipartitions [19,20]. Unlike in the bipartite case, where at most two Schmidt coefficients can be non-zero, an infinite number of Schmidt coefficients can contribute in the multipartite setting. This richer structure necessitates the development of specialized tools and measures for classifying and detecting tripartite and higher-order entanglement [21–23]. Several experimental platforms can realize genuine three-body entanglement, including trapped-ion crystals [24–26], superconducting qubits [27,28], cavity and circuit QED systems [29–31], and optomechanical architectures [32–34]. Despite this progress, a detailed understanding of how specific model Hamiltonians generate distinct forms of multipartite entanglement remains an important open problem.

In response to the previous question, we show how tripartite entanglement can be generated by modeling a system of three interacting particles as three coupled quantum harmonic oscillators. This is achieved by decomposing the total wavefunction into a superposition of single-particle states through Schmidt-mode expansions. Our analysis reveals how the resulting entanglement depends on the mixing angles induced by the couplings, thereby extending the work of Makarov [16], who studied entanglement in two coupled harmonic oscillators controlled by a single external parameter. Building on our previous work [35], we first derive the energy spectrum of the system. We then express the total wavefunction as a combination of individual particle wavefunctions parameterized by Schmidt coefficients. These coefficients can be obtained explicitly in terms of the Exton hypergeometric function K_{16} after a sequence of algebraic manipulations. We show that the Schmidt coefficients and consequently the entanglement properties, depend on the type of interaction between the particles, corresponding to the different bipartitions (A - BC , B - AC , C - AB), leading to three distinct Schmidt-mode decompositions. For each case, we first determine the reduced density matrix and then derive the purity function. This shows that classifying states based on subsystem partition is essential for meaningfully discussing entanglement because each interaction exhibits a distinct entanglement behavior. Our numerical analysis shows the effectiveness of each interaction in producing entanglement. While some interactions may exhibit instability or reduced entanglement purity, others may offer advantages under specific conditions. By appropriately adjusting the system parameters, we can recover the key results outlined by Makarov [16]. Our results could guide future research to optimize coupling configurations in order to enhance entanglement in practical quantum systems.

It is important to note that the studied Hamiltonian is quadratic, which implies the ground state is Gaussian. The ground state and its entanglement properties are well understood, as discussed in [35]. However, the picture changes significantly when we examine the excited states. Although the Hamiltonian remains quadratic, the excited states are no longer Gaussian. This may seem surprising at first, but it is actually a direct consequence of the discrete nature of the energy spectrum. These non-Gaussian states are not only mathematically different, but they also have a much richer and more intricate structure of quantum correlations. In our study, we aim to explore these states in detail and demonstrate how their entanglement properties extend beyond the scope of standard Gaussian tools. Consequently, they require analysis with different tools and a broader perspective. Addressing these non-Gaussian properties is a technical challenge, but it is necessary for exploring quantum correlations in systems with otherwise simple dynamics.

To emphasize the practical significance of our model, it is important to note that systems of coupled harmonic modes can be realized in many modern quantum platforms. Examples include optomechanical resonators [36,37], trapped-ion chains [38,39], and superconducting oscillator networks [40,41]. In these systems, the effective couplings between modes can be engineered through electromagnetic interactions, laser-mediated forces, or tunable circuit elements. The frequencies ω_i and coupling parameters D_{ij} can be controlled with high precision in such systems, allowing for the measurement of entanglement between selected subsystems (e.g., A - BC) via state tomography or correlation functions. Our theoretical predictions are robust against moderate detunings of mode frequencies, and weak damping because the primary effect on entanglement arises from the relative strength of couplings rather than absolute parameter values. This robustness suggests that the non-Gaussian entanglement features and mode hybridization described in our study can be experimentally probed with current technologies.

The present paper is organized as follows. In Section 2, we consider the Hamiltonian of three coupled quantum harmonic oscillators and derive the solutions of the energy spectrum. In Section 3, we turn to the tripartite Schmidt decomposition, and show how the Schmidt coefficients are necessary for analyzing the entangled states obtained from the eigenstates. In Section 4, we explore entanglement in our system, starting with the general case and progressing to different interaction scenarios. In Section 5, we present several examples and plots, accompanied by discussions, to illustrate the results obtained from our analysis. In Section 6, we demonstrate the non-Gaussianity using two approaches: moment analysis and Wigner function negativity. In Appendix A, we treat a special case of two interacting particles. Finally, we summarize and conclude our findings.

2. Hamiltonian formalism

We consider a stable system of three interacting particles, described by a Hamiltonian representing three coupled harmonic oscillators with different masses m_i and frequencies ω_i ($i = 1, 2, 3$). This is

$$H = \sum_{i=1}^3 \left(\frac{p_i^2}{2m_i} + \frac{1}{2} m_i \omega_i^2 x_i^2 \right) + \sum_{i \neq j=1}^3 D_{ij} x_i x_j, \quad (1)$$

where p_i and x_i represent the momentum and position operators of each oscillator, respectively, while the coupling terms D_{ij} describe the interaction between neighboring oscillators. To clarify the physical content of the model, we note that the Hamiltonian (1) represents a generic system of three coupled harmonic modes. This description is widely used as an effective model for various physical systems, such as mechanical resonators, vibrational modes in molecular systems, optical or microwave cavity modes, and effective bosonic excitations in solid-state or superconducting circuits [36,42,43]. In this framework, the frequencies ω_i correspond to

the bare mode frequencies, and the coupling parameters D_{ij} (or equivalently J_{ij}) characterize bilinear interactions between different modes, which may originate from elastic, electromagnetic, or mediated interactions.

Rescaling the phase-space variables and subsequently diagonalizing them using the $SU(3)$ algebra provides an exact and systematic way to decouple the interacting modes into independent normal modes with frequencies Σ_i . This algebraic approach was developed and discussed in detail in [35]. The mixing angles $(\theta_1, \theta_2, \theta_3)$ naturally arise from this diagonalization procedure and encode the hybridization of the original modes induced by the couplings. Physically, these angles depend on the relative strengths of the couplings and the frequency detunings. They can be indirectly tuned in realistic systems by adjusting interaction strengths, external fields, or geometric configurations. This general formulation allows us to analyze entanglement distribution and mode correlations in a broad class of multipartite continuous-variable systems. Indeed, we can perform a transformation to write the Hamiltonian (1) as

$$H = \frac{1}{2m}(P_1^2 + P_2^2 + P_3^2) + \frac{m}{2}(\omega_1^2 X_1^2 + \omega_2^2 X_2^2 + \omega_3^2 X_3^2) + m(J_{12}X_1X_2 + J_{13}X_1X_3 + J_{23}X_2X_3), \quad (2)$$

where the new phase space variables are given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \mu_1^{-1} X_1 \\ \mu_2^{-1} X_2 \\ \mu_3^{-1} X_3 \end{pmatrix}, \quad \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} \mu_1 P_1 \\ \mu_2 P_2 \\ \mu_3 P_3 \end{pmatrix}, \quad (3)$$

and we have set

$$m = (m_1 m_2 m_3)^{\frac{1}{3}}, \quad \mu_i = \left(\frac{m_i}{m}\right)^{\frac{1}{2}}, \quad \mu_1 \mu_2 \mu_3 = 1, \quad J_{ij} = \frac{D_{ij}}{2\sqrt{m_i m_j}}, \quad (4)$$

with the indices $i, j = 1, 2, 3$. Because the Hamiltonian H (2) contains interaction terms, directly analyzing the fundamental properties of the system presents significant challenges. However, these difficulties can be avoided by expressing the Hamiltonian in matrix form as

$$H = \frac{1}{2m} \sum_{i,j=1}^3 P_i \delta_{ij} P_j + \frac{1}{2} m \sum_{i,j=1}^3 X_i \mathcal{R}_{ij} X_j, \quad (5)$$

where the two matrices are given by

$$\mathcal{R}_{ij} = \begin{pmatrix} \omega_1^2 & J_{12} & J_{13} \\ J_{12} & \omega_2^2 & J_{23} \\ J_{13} & J_{23} & \omega_3^2 \end{pmatrix}, \quad X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}. \quad (6)$$

To diagonalize the Hamiltonian (5), we employ a faithful matrix representation based on the generators of the $SU(3)$ Lie group. First, we express the matrix $\mathcal{R}_{ij}$ in terms of the Gell-Mann matrices λ_i as

$$\mathcal{R}_{ij} = J_{12}\lambda_1 + J_{13}\lambda_4 + J_{23}\lambda_6 + \text{diag}(\omega_1^2, \omega_2^2, \omega_3^2), \quad (7)$$

and make a rotation with three angles $(\theta_1, \theta_2, \theta_3)$

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = M \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} = M \begin{pmatrix} \tilde{P}_1 \\ \tilde{P}_2 \\ \tilde{P}_3 \end{pmatrix}, \quad (8)$$

where the matrix M has the form

$$M = e^{i\theta_3 \lambda_7} e^{i\theta_2 \lambda_2} e^{i\theta_1 \lambda_5} \quad (9)$$

$$= \begin{pmatrix} \cos \theta_1 \cos \theta_2 & \sin \theta_2 & \cos \theta_2 \sin \theta_1 \\ -\sin \theta_1 \sin \theta_3 - \cos \theta_1 \cos \theta_3 \sin \theta_2 & \cos \theta_2 \cos \theta_3 & \cos \theta_1 \sin \theta_3 - \sin \theta_1 \cos \theta_3 \sin \theta_2 \\ -\sin \theta_1 \cos \theta_3 + \cos \theta_1 \sin \theta_2 \sin \theta_3 & -\cos \theta_2 \sin \theta_3 & \cos \theta_1 \cos \theta_3 + \sin \theta_1 \sin \theta_2 \sin \theta_3 \end{pmatrix}. \quad (10)$$

By employing both (7) and (10), we establish the following relation

$$\mathcal{R}_{ij} = M \text{diag}(\Sigma_1^2, \Sigma_2^2, \Sigma_3^2) M^{-1}, \quad (11)$$

and we have set the quantities

$$\omega_1^2 = (\Sigma_1^2 \cos^2 \theta_1 + \Sigma_3^2 \sin^2 \theta_1) \cos^2 \theta_2 + \Sigma_2^2 \sin^2 \theta_2, \quad (12)$$

$$\omega_2^2 = 2 \frac{(\Sigma_2^2 \cos^2 \theta_2 + (\Sigma_1^2 \cos^2 \theta_1 + \Sigma_3^2 \sin^2 \theta_1) \sin^2 \theta_2) \cos^2 \theta_3 + (\Sigma_3^2 \cos^2 \theta_1 + \Sigma_1^2 \sin^2 \theta_1) \sin^2 \theta_3}{2} + \frac{\Sigma_1^2 - \Sigma_3^2}{2} \sin 2\theta_1 \sin \theta_2 \sin 2\theta_3, \quad (13)$$

$$\omega_3^2 = 2 \frac{(\Sigma_3^2 \cos^2 \theta_1 + \Sigma_1^2 \sin^2 \theta_1) \cos^2 \theta_3 + (\Sigma_2^2 \cos^2 \theta_2 + (\Sigma_1^2 \cos^2 \theta_1 + \Sigma_3^2 \sin^2 \theta_1) \sin^2 \theta_2) \sin^2 \theta_3}{2} - \frac{\Sigma_1^2 - \Sigma_3^2}{2} \sin 2\theta_1 \sin \theta_2 \sin 2\theta_3, \quad (14)$$

$$J_{12} = -\frac{(\Sigma_1^2 - \Sigma_2^2) \cos^2 \theta_1 + (\Sigma_3^2 - \Sigma_2^2) \sin^2 \theta_1}{2} \sin 2\theta_2 \cos \theta_3 - \frac{\Sigma_1^2 - \Sigma_3^2}{2} \sin 2\theta_1 \cos \theta_2 \sin \theta_3, \quad (15)$$

$$J_{13} = \frac{(\Sigma_1^2 - \Sigma_2^2) \cos^2 \theta_1 + (\Sigma_3^2 - \Sigma_2^2) \sin^2 \theta_1}{2} \sin 2\theta_2 \sin \theta_3 - \frac{\Sigma_1^2 - \Sigma_3^2}{2} \sin 2\theta_1 \cos \theta_2 \cos \theta_3, \quad (16)$$

$$J_{23} = \frac{\Sigma_1^2 - \Sigma_3^2}{2} \sin 2\theta_1 \sin \theta_2 \cos 2\theta_3 - \frac{\left((\Sigma_2^2 - \Sigma_3^2) \cos^2 \theta_1 - (\Sigma_1^2 - \Sigma_2^2) \sin^2 \theta_1\right) \cos^2 \theta_2 + (\Sigma_1^2 - \Sigma_3^2) \cos 2\theta_1 \sin^2 \theta_2}{2} \sin 2\theta_3. \quad (17)$$

It is convenient to define the ratios between the coupling parameters J_{ij} (with $i, j = 1, 2, 3$) and the squared frequency differences $\omega_i^2 - \omega_j^2$. These ratios are given by

$$\frac{2J_{12}}{\omega_1^2 - \omega_2^2} = \frac{-(\Sigma_1^2 - \Sigma_2^2) \cos^2 \theta_1 \sin 2\theta_2 \cos \theta_3 + (\Sigma_2^2 - \Sigma_3^2) \sin^2 \theta_1 \sin 2\theta_2 \cos \theta_3 + (\Sigma_3^2 - \Sigma_1^2) \sin 2\theta_1 \cos \theta_2 \sin \theta_3}{(\Sigma_2^2 - \Sigma_3^2) (\sin^2 \theta_2 - \cos^2 \theta_2 \cos^2 \theta_3) + (\Sigma_3^2 - \Sigma_1^2) (\sin^2 \theta_1 \sin^2 \theta_3 - \cos^2 \theta_1 \cos^2 \theta_2 + \cos^2 \theta_1 \sin^2 \theta_2 \cos^2 \theta_3 + \frac{1}{2} \sin 2\theta_1 \sin \theta_2 \sin 2\theta_3)}, \quad (18)$$

$$\frac{2J_{13}}{\omega_1^2 - \omega_3^2} = \frac{(\Sigma_1^2 - \Sigma_2^2) \cos^2 \theta_1 \sin 2\theta_2 \sin \theta_3 - (\Sigma_2^2 - \Sigma_3^2) \sin^2 \theta_1 \sin 2\theta_2 \sin \theta_3 + (\Sigma_3^2 - \Sigma_1^2) \sin 2\theta_1 \cos \theta_2 \sin \theta_3}{(\Sigma_3^2 - \Sigma_1^2) (\sin^2 \theta_1 \cos^2 \theta_3 - \cos^2 \theta_1 \cos^2 \theta_2 + \cos^2 \theta_1 \sin^2 \theta_2 \sin^2 \theta_3 - \frac{1}{2} \sin 2\theta_1 \sin \theta_2 \sin 2\theta_3) + (\Sigma_2^2 - \Sigma_3^2) (\sin^2 \theta_2 - \cos^2 \theta_2 \sin^2 \theta_2)}, \quad (19)$$

$$\frac{2J_{23}}{\omega_2^2 - \omega_3^2} = \frac{(\Sigma_3^2 - \Sigma_1^2) (\cos 2\theta_1 \sin^2 \theta_2 \sin 2\theta_3 - \sin 2\theta_1 \sin \theta_2 \cos 2\theta_3) - (\Sigma_2^2 - \Sigma_3^2) \cos^2 \theta_1 \cos^2 \theta_2 \sin 2\theta_3 + (\Sigma_1^2 - \Sigma_2^2) \sin^2 \theta_1 \cos^2 \theta_2 \sin 2\theta_3}{(\Sigma_2^2 - \Sigma_3^2) \cos^2 \theta_2 \cos 2\theta_3 + (\Sigma_3^2 - \Sigma_1^2) (\sin^2 \theta_1 \cos 2\theta_3 - \cos^2 \theta_1 \sin^2 \theta_2 \cos 2\theta_3 - \sin 2\theta_1 \sin \theta_2 \sin 2\theta_3)}. \quad (20)$$

Using the above results, we transform the Hamiltonian (5) into

$$H = \frac{1}{2m} (\tilde{P}_1^2 + \tilde{P}_2^2 + \tilde{P}_3^2) + \frac{m}{2} (\Sigma_1^2 y_1^2 + \Sigma_2^2 y_2^2 + \Sigma_3^2 y_3^2), \quad (21)$$

which describes three decoupled harmonic oscillators and, therefore, the energy spectrum can be easily obtained. Indeed, solving the eigenvalue equation

$$H |n_1, n_2, n_3\rangle = E_{n_1, n_2, n_3} |n_1, n_2, n_3\rangle, \quad (22)$$

we end up with the eigenvalues

$$E_{n_1, n_2, n_3} = \hbar \left(\Sigma_1 n_1 + \Sigma_2 n_2 + \Sigma_3 n_3 + \frac{\Sigma_1 + \Sigma_2 + \Sigma_3}{2} \right), \quad (23)$$

and the normalized wavefunctions

$$\psi_{n_1, n_2, n_3}(x_1, x_2, x_3) = \left(\frac{m\omega}{\pi \hbar} \right)^{\frac{3}{4}} \frac{e^{-\frac{m}{2\hbar} (\Sigma_1 y_1^2 + \Sigma_2 y_2^2 + \Sigma_3 y_3^2)}}{\sqrt{2^{n_1+n_2+n_3} n_1! n_2! n_3!}} H_{n_1} \left(\sqrt{\frac{m\Sigma_1}{\hbar}} y_1 \right) H_{n_2} \left(\sqrt{\frac{m\Sigma_2}{\hbar}} y_2 \right) H_{n_3} \left(\sqrt{\frac{m\Sigma_3}{\hbar}} y_3 \right), \quad (24)$$

where n_1, n_2, n_3 are integers and H_{n_i} ($i = 1, 2, 3$) denote the Hermite polynomials. Equation (24) explicitly represents a tensor product of three independent solutions, each corresponding to a one-dimensional harmonic oscillator. To rewrite (24) in terms of the original variables (x_1, x_2, x_3) , we apply the inverse transformations of (8) to express the new coordinates (y_1, y_2, y_3) as

$$y_1 = \mu_1 \cos \theta_1 \cos \theta_2 x_1 - \mu_2 (\sin \theta_1 \sin \theta_3 + \cos \theta_1 \cos \theta_3 \sin \theta_2) x_2 - \mu_3 (\sin \theta_1 \cos \theta_3 - \cos \theta_1 \sin \theta_2 \sin \theta_3) x_3, \quad (25)$$

$$y_2 = \mu_1 \sin \theta_2 x_1 + \mu_2 \cos \theta_2 \cos \theta_3 x_2 - \mu_3 \cos \theta_2 \sin \theta_3 x_3, \quad (26)$$

$$y_3 = \mu_1 \cos \theta_2 \sin \theta_1 x_1 + \mu_2 (\cos \theta_1 \sin \theta_3 - \sin \theta_1 \cos \theta_3 \sin \theta_2) x_2 + \mu_3 (\cos \theta_1 \cos \theta_3 + \sin \theta_1 \sin \theta_2 \sin \theta_3) x_3, \quad (27)$$

which can then be substituted into (24) to obtain the exact wavefunctions $\psi_{n_1, n_2, n_3}(x_1, x_2, x_3)$, representing the solutions of the three coupled harmonic oscillators with quadratic interactions described by the Hamiltonian (1).

To investigate the entanglement properties of our system using Schmidt modes, we consider a special limiting case. This scenario arises when the coupling parameters J_{ij} are sufficiently small, i.e., $J_{ij} \ll 1$. This condition can be realized by configuring the system appropriately, such as

$$J_{ij} = \epsilon_{ij} \tilde{J}_{ij}, \quad (28)$$

where the parameter ϵ_{ij} is chosen such that $\epsilon_{ij} \rightarrow 0$. We also assume that the oscillator frequencies ω_i are comparable to each other and of the same order of magnitude as the coupling parameters J_{ij} . Then, we can write

$$\omega_i^2 - \omega_j^2 = \epsilon_{ij} (\tilde{\omega}_i^2 - \tilde{\omega}_j^2). \quad (29)$$

According to (12-14), a similar relation can be established for the new frequencies

$$\Sigma_i^2 - \Sigma_j^2 = \epsilon_{ij} (\tilde{\Sigma}_i^2 - \tilde{\Sigma}_j^2). \quad (30)$$

To simplify our calculations, let us consider the case where $\epsilon_{12} = \epsilon_{23} = \epsilon_{13}$. Under this assumption, we can establish the following ratios

$$\frac{2\tilde{J}_{12}}{\tilde{\omega}_1^2 - \tilde{\omega}_2^2} = \frac{-(\tilde{\Sigma}_1^2 - \tilde{\Sigma}_2^2) \cos^2 \theta \sin 2\phi \cos \varphi + (\tilde{\Sigma}_2^2 - \tilde{\Sigma}_3^2) \sin^2 \theta \sin 2\phi \cos \varphi + (\tilde{\Sigma}_3^2 - \tilde{\Sigma}_1^2) \sin 2\theta \cos \phi \sin \varphi}{(\tilde{\Sigma}_2^2 - \tilde{\Sigma}_3^2) (\sin^2 \phi - \cos^2 \phi \cos^2 \varphi) + (\tilde{\Sigma}_3^2 - \tilde{\Sigma}_1^2) (\sin^2 \theta \sin^2 \varphi - \cos^2 \theta \cos^2 \phi + \cos^2 \theta \sin^2 \phi \cos^2 \varphi + \frac{1}{2} \sin 2\theta \sin \phi \sin 2\varphi)}, \quad (31)$$

$$\frac{2\tilde{J}_{13}}{\tilde{\omega}_1^2 - \tilde{\omega}_3^2} = \frac{(\tilde{\Sigma}_1^2 - \tilde{\Sigma}_2^2) \cos^2 \theta \sin 2\phi \sin \varphi - (\tilde{\Sigma}_2^2 - \tilde{\Sigma}_3^2) \sin^2 \theta \sin 2\phi \sin \varphi + (\tilde{\Sigma}_3^2 - \tilde{\Sigma}_1^2) \sin 2\theta \cos \phi \cos \varphi}{(\tilde{\Sigma}_3^2 - \tilde{\Sigma}_1^2) (\sin^2 \theta \cos^2 \varphi - \cos^2 \theta \cos^2 \phi + \cos^2 \theta \sin^2 \phi \sin^2 \varphi - \frac{1}{2} \sin 2\theta \sin \phi \sin 2\varphi) + (\tilde{\Sigma}_2^2 - \tilde{\Sigma}_3^2) (\sin^2 \phi - \cos^2 \phi \sin^2 \varphi)}, \quad (32)$$

$$\frac{2\tilde{J}_{23}}{\tilde{\omega}_2^2 - \tilde{\omega}_3^2} = \frac{(\tilde{\Sigma}_3^2 - \tilde{\Sigma}_1^2) (\cos 2\theta \sin^2 \phi \sin 2\varphi - \sin 2\theta \sin \phi \cos 2\varphi) - (\tilde{\Sigma}_2^2 - \tilde{\Sigma}_3^2) \cos^2 \theta \cos^2 \phi \sin 2\varphi + (\tilde{\Sigma}_1^2 - \tilde{\Sigma}_2^2) \sin^2 \theta \cos^2 \phi \sin 2\varphi}{(\tilde{\Sigma}_2^2 - \tilde{\Sigma}_3^2) \cos^2 \phi \cos 2\varphi + (\tilde{\Sigma}_3^2 - \tilde{\Sigma}_1^2) (\sin^2 \theta \cos 2\varphi - \cos^2 \theta \sin^2 \phi \cos 2\varphi - \sin 2\theta \sin \phi \sin 2\varphi)}. \quad (33)$$

Considering all the above requirements, we can express the wavefunctions (24) of the system as

$$\psi_{n_1, n_2, n_3}(x_1, x_2, x_3) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{3}{4}} \frac{e^{-\frac{m\omega}{2\hbar}(q_1^2+q_2^2+q_3^2)}}{\sqrt{2^{n_1+n_2+n_3}n_1!n_2!n_3!}} H_{n_1}\left(\sqrt{\frac{m\omega}{\hbar}}q_1\right) H_{n_2}\left(\sqrt{\frac{m\omega}{\hbar}}q_2\right) H_{n_3}\left(\sqrt{\frac{m\omega}{\hbar}}q_3\right), \quad (34)$$

where the new variables q_i have the same forms as in (25–27) except that the angles $(\vartheta_1, \vartheta_2, \vartheta_3)$ are replaced by (θ, ϕ, φ) , respectively, namely

$$q_1 = \mu_1 \cos \theta \cos \phi x_1 - \mu_2 (\sin \theta \sin \varphi + \cos \theta \cos \varphi \sin \phi) x_2 - \mu_3 (\sin \theta \cos \varphi - \cos \theta \sin \phi \sin \varphi) x_3, \quad (35)$$

$$q_2 = \mu_1 \sin \phi x_1 + \mu_2 \cos \phi \cos \varphi x_2 - \mu_3 \cos \phi \sin \varphi x_3, \quad (36)$$

$$q_3 = \mu_1 \cos \phi \sin \varphi x_1 + \mu_2 (\cos \theta \sin \varphi - \sin \theta \cos \phi \sin \varphi) x_2 + \mu_3 (\cos \theta \cos \varphi + \sin \theta \sin \phi \sin \varphi) x_3. \quad (37)$$

The corresponding eigenvalues have the forms

$$E_{n_1, n_2, n_3} = \hbar\omega \left(n_1 + n_2 + n_3 + \frac{3}{2}\right), \quad (38)$$

and we set the frequency $\omega = (\omega_1\omega_2\omega_3)^{\frac{1}{3}}$. The established mathematical tools can serve as a starting point to perform a Schmidt decomposition of the above wavefunctions. This decomposition will allow us to quantify and analyze the entanglement between the oscillators based on the Schmidt modes.

3. Schmidt decomposition

We emphasize that the ground-state wavefunction is disentangled due to the separability in the zero mode. To proceed further, let us apply the Schmidt decomposition to express the wavefunctions (34) as

$$\psi_{n_1, n_2, n_3}^{ABC}(x_1, x_2, x_3) = \sum_{l, k, m=0}^{\infty} A_{n_1, n_2, n_3}^{k, l, m} \varphi_k^A(x_1) \phi_l^B(x_2) \chi_m^C(x_3), \quad (39)$$

where $\varphi_k^A(x_1)$, $\phi_l^B(x_2)$ and $\chi_m^C(x_3)$ have the forms

$$\varphi_k^A(x_1) = \left(\frac{\sqrt{\frac{m\omega}{\pi\hbar}}}{2^k k!}\right)^{\frac{1}{2}} e^{-\frac{m\omega}{2\hbar}\mu_1^2 x_1^2} H_k^A\left(\sqrt{\frac{m\omega}{\hbar}}\mu_1 x_1\right), \quad (40)$$

$$\phi_l^B(x_2) = \left(\frac{\sqrt{\frac{m\omega}{\pi\hbar}}}{2^l l!}\right)^{\frac{1}{2}} e^{-\frac{m\omega}{2\hbar}\mu_2^2 x_2^2} H_l^B\left(\sqrt{\frac{m\omega}{\hbar}}\mu_2 x_2\right), \quad (41)$$

$$\chi_m^C(x_3) = \left(\frac{\sqrt{\frac{m\omega}{2\pi\hbar}}}{2^m m!}\right)^{\frac{1}{2}} e^{-\frac{m\omega}{2\hbar}\mu_3^2 x_3^2} H_m^C\left(\sqrt{\frac{m\omega}{\hbar}}\mu_3 x_3\right), \quad (42)$$

and the Schmidt coefficients can be obtained from the orthogonality conditions as

$$A_{n_1, n_2, n_3}^{k, l, m} = \iiint \psi_{n_1, n_2, n_3}^{ABC}(x_1, x_2, x_3) \varphi_k^A(x_1) \phi_l^B(x_2) \chi_m^C(x_3) dx_1 dx_2 dx_3, \quad (43)$$

or equivalent after substitution

$$A_{n_1, n_2, n_3}^{k, l, m} = \frac{\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{3}{2}}}{\sqrt{2^{n_1+n_2+n_3+k+l+m}n_1!n_2!n_3!k!l!m!}} \iiint dx_1 dx_2 dx_3 e^{-\frac{m\omega}{2\hbar}(q_1^2+q_2^2+q_3^2+\mu_1^2 x_1^2+\mu_2^2 x_2^2+\mu_3^2 x_3^2)} H_{n_1}\left(\sqrt{\frac{m\omega}{\hbar}}q_1\right) H_{n_2}\left(\sqrt{\frac{m\omega}{\hbar}}q_2\right) H_{n_3}\left(\sqrt{\frac{m\omega}{\hbar}}q_3\right) H_k^A\left(\sqrt{\frac{m\omega}{\hbar}}\mu_1 x_1\right) H_l^B\left(\sqrt{\frac{m\omega}{\hbar}}\mu_2 x_2\right) H_m^C\left(\sqrt{\frac{m\omega}{\hbar}}\mu_3 x_3\right). \quad (44)$$

To explicitly obtain $A_{n_1, n_2, n_3}^{k, l, m}$, we use the Rodrigues formula for the Hermite polynomials, which is

$$H_n(\omega x) = \frac{d^n}{dt^n} e^{-t^2+2\omega x t} \Big|_{t=0}, \quad (45)$$

and then (44) becomes

$$A_{n_1, n_2, n_3}^{k, l, m} = \frac{\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{3}{2}}}{\sqrt{2^{n_1+n_2+n_3+k+l+m}n_1!n_2!n_3!k!l!m!}} \iiint \frac{d^{n_1}}{dx^{n_1}} \frac{d^{n_2}}{dy^{n_2}} \frac{d^{n_3}}{dz^{n_3}} \frac{d^k}{dt^k} \frac{d^l}{ds^l} \frac{d^m}{du^m} \left\{ e^{-\frac{m\omega}{\hbar}(\mu_1^2 x_1^2+\mu_2^2 x_2^2+\mu_3^2 x_3^2)} e^{-x^2+2\sqrt{\frac{m\omega}{\hbar}}[\mu_1 \cos \theta \cos \phi x_1 - \mu_2 (\sin \theta \sin \varphi + \cos \theta \cos \varphi \sin \phi) x_2 - \mu_3 (\sin \theta \cos \varphi - \cos \theta \sin \phi \sin \varphi) x_3]} x \right.$$

$$\begin{aligned}
& e^{-y^2+2\sqrt{\frac{m\omega}{h}}[\mu_1 \sin \phi x_1 + \mu_2 \cos \phi \cos \varphi x_2 - \mu_3 \cos \phi \sin \varphi x_3]y} \\
& e^{-z^2+2\sqrt{\frac{m\omega}{h}}[\mu_1 \cos \phi \sin \theta x_1 + \mu_2 (\cos \theta \sin \varphi - \sin \theta \cos \varphi \sin \phi)x_2 + \mu_3 (\cos \theta \cos \varphi + \sin \theta \sin \phi \sin \varphi)x_3]z} \\
& e^{-t^2+2\sqrt{\frac{m\omega}{h}}\mu_1 x_1 t} e^{-s^2+2\sqrt{\frac{m\omega}{h}}\mu_2 x_2 s} e^{-w^2+2\sqrt{\frac{m\omega}{h}}\mu_3 x_3 w} \Bigg|_{x,y,z,t,s,w=0} dx_1 dx_2 dx_3.
\end{aligned} \quad (46)$$

After performing the integration over the variables x_1, x_2, x_3 , we get

$$A_{n_1, n_2, n_3}^{k, l, m} = \frac{d^{n_1}}{dx_1^{n_1}} \frac{d^{n_2}}{dy^{n_2}} \frac{d^{n_3}}{dz^{n_3}} \frac{d^k}{dt^k} \frac{d^l}{ds^l} \frac{d^m}{dw^m} \frac{e^{2a_1 tx} e^{2b_1 ty} e^{2c_1 tz} e^{2a_2 sx} e^{2b_2 sy} e^{2c_2 sz} e^{2a_3 wx} e^{2b_3 wy} e^{2c_3 wz}}{\sqrt{2^{n_1+n_2+n_3+k+l+m} n_1! n_2! n_3! k! l! m!}} \Bigg|_{x,y,z,t,s,w=0}, \quad (47)$$

where we have set the following parameters

$$a_1 = \cos \theta \cos \phi, \quad (48)$$

$$b_1 = \sin \phi, \quad (49)$$

$$c_1 = \cos \phi \sin \theta, \quad (50)$$

$$a_2 = -\sin \theta \sin \varphi - \cos \theta \cos \varphi \sin \phi, \quad (51)$$

$$b_2 = \cos \phi \cos \varphi, \quad (52)$$

$$c_2 = \cos \theta \sin \varphi - \sin \theta \cos \varphi \sin \phi, \quad (53)$$

$$a_3 = \cos \theta \sin \phi \sin \varphi - \sin \theta \cos \varphi, \quad (54)$$

$$b_3 = -\cos \phi \sin \varphi, \quad (55)$$

$$c_3 = \cos \theta \cos \varphi + \sin \theta \sin \phi \sin \varphi. \quad (56)$$

As we will see, the above set plays a crucial role in determining the purity functions for different configurations, depending on how the interactions between the oscillators are treated. We can rewrite (47) in a different form by expanding each exponential as a power series, which yields

$$\begin{aligned}
A_{n_1, n_2, n_3}^{k, l, m} &= \frac{d^{n_1}}{dx_1^{n_1}} \frac{d^{n_2}}{dy^{n_2}} \frac{d^{n_3}}{dz^{n_3}} \frac{d^k}{dt^k} \frac{d^l}{ds^l} \frac{d^m}{dw^m} \sum_{i_1, \dots, i_9=0}^{\infty} \left\{ \frac{t^{i_1+i_2+i_3} s^{i_4+i_5+i_6} w^{i_7+i_8+i_9} x^{i_1+i_4+i_7} y^{i_2+i_5+i_8} z^{i_3+i_6+i_9}}{\sqrt{2^{n_1+n_2+n_3+k+l+m} n_1! n_2! n_3! k! l! m!}} \right. \\
&\quad \left. \frac{(2a_1)^{i_1}}{i_1!} \frac{(2b_1)^{i_2}}{i_2!} \frac{(2c_1)^{i_3}}{i_3!} \frac{(2a_2)^{i_4}}{i_4!} \frac{(2b_2)^{i_5}}{i_5!} \frac{(2c_2)^{i_6}}{i_6!} \frac{(2a_3)^{i_7}}{i_7!} \frac{(2b_3)^{i_8}}{i_8!} \frac{(2c_3)^{i_9}}{i_9!} \right\} \Bigg|_{x,y,z,t,s,w=0}.
\end{aligned} \quad (57)$$

By using the identity $\frac{d^k}{dt^k} t^i \Big|_{t=0} = k! \delta_{k,i}$ and taking derivatives with respect to the variables (t, s, w, x, y, z) , we obtain

$$\begin{aligned}
A_{n_1, n_2, n_3}^{k, l, m} &= \sum_{i_1, \dots, i_9=0}^{\infty} \frac{\sqrt{n_1! n_2! n_3! k! l! m!} (a_3)^{n_1} (b_3)^{n_2} (c_3)^k (c_2)^l (c_3)^{n_3-k-l}}{(n_3-k-l+i_1+i_2+i_3+i_4)!} \frac{\left(\frac{a_2 c_3}{c_2 a_3}\right)^{i_3} \left(\frac{a_1 c_3}{c_1 a_3}\right)^{i_1} \left(\frac{b_1 c_3}{c_1 b_3}\right)^{i_2} \left(\frac{b_2 c_3}{c_2 b_3}\right)^{i_4}}{(n_2-i_2-i_4)! (l-i_3-i_4)! (k-i_1-i_2)! (n_1-i_1-i_3)! i_1! i_2! i_3! i_4!} \\
&\quad \delta_{i_1+i_2+i_3, k} \delta_{i_4+i_5+i_6, l} \delta_{i_7+i_8+i_9, m} \delta_{i_1+i_4+i_7, n_1} \delta_{i_2+i_5+i_8, n_2} \delta_{i_3+i_6+i_9, n_3}.
\end{aligned} \quad (58)$$

It is convenient to introduce Exton's hypergeometric function, defined as [44]

$$K_{16}(\alpha_1, \alpha_2, \alpha_3, \alpha_4; \beta; x, y, z, t) = \sum_{m_1, m_2, m_3, m_4=0}^{\infty} \frac{(\alpha_1)_{m_1+m_2} (\alpha_2)_{m_2+m_3} (\alpha_3)_{m_3+m_4} (\alpha_4)_{m_4+m_1}}{(\beta)_{m_1+m_2+m_3+m_4}} \frac{x^{m_1} y^{m_2} z^{m_3} t^{m_4}}{m_1! m_2! m_3! m_4!}, \quad (59)$$

where $(a)_k$ is the Pochhammer symbol $(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = (-1)^k \frac{\Gamma(1-a)}{\Gamma(1-a-k)}$. As a result, we show that (58) can be expressed in terms of K_{16} as

$$\begin{aligned}
A_{n_1, n_2, n_3}^{k, l, m} &= \delta_{m, n_1+n_2+n_3-k-l} \frac{\sqrt{n_1! n_2! n_3! k! l! (n_1+n_2+n_3-k-l)!} (a_3)^{n_1} (b_3)^{n_2} (c_1)^k (c_2)^l (c_3)^{n_3-k-l}}{(n_3-k-l)! n_1! k! n_2! l!} \\
&\quad K_{16}\left(-n_1, -k, -n_2, -l; n_3 - k - l + 1; \frac{a_2 c_3}{c_2 a_3}, \frac{a_1 c_3}{c_1 a_3}, \frac{b_1 c_3}{c_1 b_3}, \frac{b_2 c_3}{c_2 b_3}\right),
\end{aligned} \quad (60)$$

which implies that to avoid vanishing coefficients, the condition $m + k + l = n_1 + n_2 + n_3$ must be satisfied. Here, K_{16} is a four-variable polynomial of order n_1, k, n_2, l . By summing over m , we obtain the Schmidt coefficients

$$A_{n_1, n_2, n_3}^{k, l} = \sum_m A_{n_1, n_2, n_3}^{k, l, m} \delta_{m, n_1+n_2+n_3-k-l} \quad (61)$$

$$\begin{aligned}
&= \frac{\sqrt{n_3! k! l! (n_1+n_2+n_3-k-l)!} (a_3)^{n_1} (b_3)^{n_2} (c_1)^k (c_2)^l (c_3)^{n_3-k-l}}{(n_3-k-l)! \sqrt{n_1! n_2! k! l!}} \\
&\quad K_{16}\left(-n_1, -k, -n_2, -l; n_3 - k - l + 1; \frac{a_2 c_3}{c_2 a_3}, \frac{a_1 c_3}{c_1 a_3}, \frac{b_1 c_3}{c_1 b_3}, \frac{b_2 c_3}{c_2 b_3}\right).
\end{aligned} \quad (62)$$

Substituting this into (39) to write the wavefunctions as

$$\psi_{n_1, n_2, n_3}^{ABC}(x_1, x_2, x_3) = \sum_{k, l} A_{n_1, n_2, n_3}^{k, l} \varphi_k^A(x_1) \phi_l^B(x_2) \chi_{n_1+n_2+n_3-k-l}^C(x_3). \quad (63)$$

Having derived (63) together with (62), we now proceed to explore their applications. In particular, we will use the associated density matrices to determine the purity functions and investigate the fundamental properties of entanglement.

4. Entanglement analysis

We begin our analysis of entanglement in the present system by first considering the general case and then examining more specific configurations. This combined approach allows us to clearly demonstrate our results and highlight the fundamental properties that depend on the underlying physical parameters. By analyzing both broad and particular scenarios, we illustrate how entanglement is influenced by different conditions within the system and identify the key features that emerge under various parametric regimes. This progression from the general framework to specific examples provides deeper insight into how the nature of entanglement evolves with the dynamical settings governing the system.

4.1. Generic case

There are three distinct ways to construct the Schmidt decomposition of the wavefunctions (63), depending on the interaction configuration considered [45]:

Interaction	Separated subsystem	Common subsystem
A-BC	A	BC
B-AC	B	AC
C-AB	C	AB

From this table we observe that there are three distinct ways to apply the Schmidt decomposition, i.e., the same wavefunctions (63) can be expressed in different forms. This flexibility allows for alternative representations of the entanglement structure of the system. These are given by

$$\psi_{n_1, n_2, n_3}^{ABC}(x_1, x_2, x_3) = \sum_{k=0}^{n_1+n_2+n_3} \sqrt{\alpha_k} \varphi_k^A(x_1) \Theta_k^{BC}(x_2, x_3) \quad (64)$$

$$= \sum_{l=0}^{n_1+n_2+n_3} \sqrt{\beta_l} \phi_l^B(x_2) \Phi_l^{AC}(x_1, x_3) \quad (65)$$

$$= \sum_{\sigma=0}^{n_1+n_2+n_3} \sqrt{\gamma_\sigma} \chi_\sigma^C(x_3) \Xi_\sigma^{AB}(x_1, x_2), \quad (66)$$

where the sets $\{\varphi_k^A(x_1)\}$, $\{\phi_l^B(x_2)\}$, and $\{\chi_\sigma^C(x_3)\}$ consist of orthonormal pairs of single-particle states, while the sets $\{\Theta_k^{BC}(x_2, x_3)\}$, $\{\Phi_l^{AC}(x_1, x_3)\}$, and $\{\Xi_\sigma^{AB}(x_1, x_2)\}$ contain orthonormal pairs of two-particle states, defined as

$$\Theta_k^{BC}(x_2, x_3) = \sum_{l=0}^{n_1+n_2+n_3-k} \phi_l(x_2) \chi_{n_1+n_2+n_3-k-l}(x_3), \quad (67)$$

$$\Phi_l^{AC}(x_1, x_3) = \sum_{k=0}^{n_1+n_2+n_3-l} \varphi_k(x_1) \chi_{n_1+n_2+n_3-k-l}(x_3), \quad (68)$$

$$\Xi_\sigma^{AB}(x_1, x_2) = \sum_{k=0}^{n_1+n_2+n_3-\sigma} \varphi_k(x_1) \phi_{n_1+n_2+n_3-k-\sigma}(x_2), \quad (69)$$

while the parameters $\alpha_k \equiv \alpha_k^{n_1, n_2, n_3}$, $\beta_l \equiv \beta_l^{n_1, n_2, n_3}$, and $\gamma_\sigma \equiv \gamma_\sigma^{n_1, n_2, n_3}$ depend on the Schmidt modes and take the forms

$$\alpha_k = \sum_{l=0}^{n_1+n_2+n_3-k} \left(A_{n_1, n_2, n_3}^{k, l} \right)^2, \quad (70)$$

$$\beta_l = \sum_{k=0}^{n_1+n_2+n_3-l} \left(A_{n_1, n_2, n_3}^{k, l} \right)^2, \quad (71)$$

$$\gamma_\sigma = \sum_{k=0}^{n_1+n_2+n_3-\sigma} \left(A_{n_1, n_2, n_3}^{k, n_1+n_2+n_3-k-\sigma} \right)^2, \quad (72)$$

and they are simply the eigenvalues of the reduced density matrices (ρ^A, ρ^{BC}) , (ρ^B, ρ^{AC}) , and (ρ^C, ρ^{AB}) , respectively, defined as

$$\rho^A = \sum_{k=0}^{n_1+n_2+n_3} \alpha_k \varphi_k^A(x_1) \varphi_k^{*A}(x'_1), \quad \rho^{BC} = \sum_{k=0}^{n_1+n_2+n_3} \alpha_k \Theta_k^{BC}(x_2, x_3) \Theta_k^{*BC}(x'_2, x'_3), \quad (73)$$

$$\rho^B = \sum_{l=0}^{n_1+n_2+n_3} \beta_l \phi_l^B(x_2) \phi_l^{*B}(x'_2), \quad \rho^{AC} = \sum_{l=0}^{n_1+n_2+n_3} \beta_l \Phi_l^{AC}(x_1, x_3) \Phi_l^{*AC}(x'_1, x'_3), \quad (74)$$

$$\rho^C = \sum_{\sigma=0}^{n_1+n_2+n_3} \gamma_\sigma \chi_\sigma^C(x_3) \chi_\sigma^{*C}(x'_3), \quad \rho^{AB} = \sum_{\sigma=0}^{n_1+n_2+n_3} \gamma_\sigma \Xi_\sigma^{AB}(x_1, x_2) \Xi_\sigma^{*AB}(x'_1, x'_2). \quad (75)$$

From these densities, we can derive the corresponding purity functions $P_{n_1, n_2, n_3}^A(\theta, \varphi, \phi)$, $P_{n_1, n_2, n_3}^B(\theta, \varphi, \phi)$, and $P_{n_1, n_2, n_3}^C(\theta, \varphi, \phi)$. This constitutes the focus of the subsequent analysis, in which we examine various configurations the quantum numbers (n_1, n_2, n_3) , with a particular emphasis on entanglement as the central aspect of the investigation.

4.2. A-BC interaction

4.2.1. Purity analysis

We first consider the A-BC configuration, described by the Schmidt decomposition given in (64), which reads

$$\psi_{n_1, n_2, n_3}^{ABC}(x_1, x_2, x_3)_{ZZ} = \sum_{k=0}^{n_1+n_2+n_3} \sqrt{\alpha_k} \varphi_k^A(x_1) \Theta_k^{BC}(x_2, x_3). \quad (76)$$

We then analyze three distinct scenarios: $(n_1 = 0, n_2 = 0, n_3 \neq 0)$, $(n_1 = 0, n_2 \neq 0, n_3 = 0)$, and $(n_1 \neq 0, n_2 = 0, n_3 = 0)$, with the aim of identifying their primary differences.

A₁. $n_1 = 0, n_2 = 0, n_3 \neq 0$: After setting $n_1 = 0, n_2 = 0$ in (70) and using (62) together with (50), (53), (56), we get the Schmidt modes

$$\alpha_k^{0,0,n_3} = \frac{n_3!}{k!(n_3-k)!} \sin^{2k} \theta \cos^{2k} \phi (1 - \sin^2 \theta \cos^2 \phi)^{n_3-k}, \quad (77)$$

which allows us to verify the von Neumann normalization condition

$$\text{tr} \rho_{0,0,n_3}^A = \text{tr} \rho_{0,0,n_3}^{BC} = \sum_{k=0}^{n_3} \alpha_k^{0,0,n_3} = 1. \quad (78)$$

In general, the purity function is defined as the trace of the squared density operator. Since $P_{n_1, n_2, n_3}^A(\theta, \varphi, \phi) = P_{n_1, n_2, n_3}^{BC}(\theta, \varphi, \phi)$, we can write

$$P_{n_1, n_2, n_3}^A(\theta, \varphi, \phi) = \text{tr}(\rho_{n_1, n_2, n_3}^A)^2 = \sum_{k=0}^{n_1+n_2+n_3} (\alpha_k^{n_1, n_2, n_3})^2. \quad (79)$$

For the configuration $n_1 = n_2 = 0, n_3 \neq 0$, we obtain

$$P_{0,0,n_3}^A(\theta, \varphi, \phi) = \sum_{k=0}^{n_3} \left(\frac{n_3!}{k!(n_3-k)!} \right)^2 c_1^{4k} (c_3^2 + c_2^2)^{2(n_3-k)} \quad (80)$$

$$= \left((\cos^2 \theta + \sin^2 \theta \sin^2 \phi)^2 - \sin^4 \theta \cos^4 \phi \right)^{n_3} P_{n_3} \left(\frac{(\cos^2 \theta + \sin^2 \theta \sin^2 \phi)^2 + \sin^4 \theta \cos^4 \phi}{(\cos^2 \theta + \sin^2 \theta \sin^2 \phi)^2 - \sin^4 \theta \cos^4 \phi} \right), \quad (81)$$

where we have used the following identity [46]

$$\sum_{k=0}^n \left(\frac{n!}{k!(n-k)!} \right)^2 x^k = (1-x)^n P_n \left(\frac{1+x}{1-x} \right), \quad (82)$$

and P_n denote the Legendre polynomials of order n . The first four polynomials are given by

$$P_0(z) = 1, \quad P_1(z) = z, \quad P_2(z) = \frac{1}{2}(3z^2 - 1), \quad P_3(z) = \frac{1}{2}(5z^3 - 3z). \quad (83)$$

A₂. $n_1 = 0, n_2 \neq 0, n_3 = 0$: As before, we consider (70) under the conditions $n_1 = 0, n_3 = 0$, together with (62), (49), (52), and (55), to obtain the Schmidt modes

$$\alpha_k^{0,n_2,0} = \sum_{l=0}^{n_2-k} \left(A_{0,n_2,0}^{k,l} \right)^2 = \frac{n_2! (b_1^2)^k}{k! (n_2-k)!} (b_3^2 + b_2^2)^{n_2-k}, \quad (84)$$

showing the von Neumann normalization condition

$$\text{tr} \rho_{0,n_2,0}^A = \text{tr} \rho_{0,n_2,0}^{BC} = \sum_{k=0}^{n_2} \alpha_k^{0,n_2,0} = (b_1^2 + b_3^2 + b_2^2)^{n_2} = 1. \quad (85)$$

From (79) with $n_1 = 0, n_3 = 0$, we obtain

$$P_{0,n_2,0}^A(\theta, \varphi, \phi) = \left((b_3^2 + b_2^2)^2 - b_1^4 \right)^{n_2} P_{n_2} \left(\frac{(b_3^2 + b_2^2)^2 + b_1^4}{(b_3^2 + b_2^2)^2 - b_1^4} \right) \quad (86)$$

$$= \cos^{n_2}(2\phi) \mathcal{P}_{n_2} \left(\frac{\cos(4\phi) + 3}{4 \cos 2\phi} \right). \quad (87)$$

For $n_2 = 1, 2$, we get

$$P_{0,1,0}^A(\theta, \varphi, \phi) = \frac{1}{4}(\cos 4\phi + 3), \quad (88)$$

$$P_{0,2,0}^A(\theta, \varphi, \phi) = \frac{5}{16} \cos 4\phi + \frac{3}{64} \cos 8\phi + \frac{41}{64}. \quad (89)$$

By comparing these purities with those obtained in (147) and (148), we can identify the factors responsible for the observed differences. Consequently, we conclude that the quantum numbers play a crucial role in producing these distinct results.

A₃. $n_1 \neq 0, n_2 = 0, n_3 = 0$: To complete our study, we analyze the final case and highlight the factors that distinguish it from the two previous ones. From (70) under the conditions $n_2 = 0$ and $n_3 = 0$, together with (62), (48), (51), and (54), we find the corresponding Schmidt modes

$$\alpha_k^{n_1,0,0} = \sum_{l=0}^{n_1-k} \left(A_{n_1,0,0}^{k,l} \right)^2 = \frac{n_1! \binom{a_1^2}{k}}{k! (n_1-k)!} (a_3^2 + a_2^2)^{n_1-k}, \quad (90)$$

which leads to the von Neumann normalization condition

$$\text{tr} \rho_{n_1,0,0}^A = \text{tr} \rho_{n_1,0,0}^{BC} = \sum_{k=0}^{n_1} \alpha_k^{n_1,0,0} = (a_1^2 + a_3^2 + a_2^2)^{n_1} = 1. \quad (91)$$

By setting $n_2 = 0$ and $n_3 = 0$ into (79), we get the following purity

$$P_{n_1,0,0}^A(\theta, \varphi, \phi) = \left((a_3^2 + a_2^2)^2 - a_1^4 \right)^{n_1} \mathcal{P}_{n_1} \left(\frac{(a_3^2 + a_2^2)^2 + a_1^4}{(a_3^2 + a_2^2)^2 - a_1^4} \right) \quad (92)$$

$$= \left((\sin^2 \theta + \cos^2 \theta \sin^2 \phi)^2 - \cos^4 \theta \cos^4 \phi \right)^{n_1} \mathcal{P}_{n_1} \left(\frac{(\sin^2 \theta + \cos^2 \theta \sin^2 \phi)^2 + \cos^4 \theta \cos^4 \phi}{(\sin^2 \theta + \cos^2 \theta \sin^2 \phi)^2 - \cos^4 \theta \cos^4 \phi} \right). \quad (93)$$

In particular, for $n_1 = 1, 2$, we have

$$P_{1,0,0}^A(\theta, \varphi, \phi) = (\sin^2 \theta + \cos^2 \theta \sin^2 \phi)^2 + \cos^4 \theta \cos^4 \phi, \quad (94)$$

$$P_{2,0,0}^A(\theta, \varphi, \phi) = \frac{1}{2} \left(3 \left((\sin^2 \theta + \cos^2 \theta \sin^2 \phi)^2 + \cos^4 \theta \cos^4 \phi \right)^2 - \left((\sin^2 \theta + \cos^2 \theta \sin^2 \phi)^2 - \cos^4 \theta \cos^4 \phi \right)^2 \right). \quad (95)$$

4.2.2. Von Neumann analysis

While purity and the von Neumann entropy are directly related for Gaussian states [43], this correspondence generally breaks down in non-Gaussian regimes. Studying both quantities is therefore essential, particularly for excited states, where their different sensitivities reveal genuinely non-Gaussian features and provide complementary insights into multipartite entanglement. The von Neumann entropy is defined as

$$S_{vN} = -\text{Tr}(\rho \ln \rho) = -\sum_k \lambda_k \ln \lambda_k, \quad (96)$$

where λ_k are the eigenvalues of the reduced density matrix ρ . For the **A-BC** bipartition, it can be written as

$$S_{vN}^{n_1, n_2, n_3} = -\sum_{k=0}^{n_1+n_2+n_3} \alpha_k^{n_1, n_2, n_3} \ln(\alpha_k^{n_1, n_2, n_3}). \quad (97)$$

We now examine the von Neumann entropy for the three configurations previously analyzed using the purity function.

A₁. $n_1 = 0, n_2 = 0, n_3 \neq 0$:

For this specific configuration, we have

$$S_{vN}^{0,0,n_3} = -\sum_{k=0}^{n_3} \binom{n_3}{k} (c_1^2)^k (c_2^2 + c_3^2)^{n_3-k} \ln \left[\binom{n_3}{k} (c_1^2)^k (c_2^2 + c_3^2)^{n_3-k} \right]. \quad (98)$$

For $n_3 = 1$, the von Neumann entropy takes the form

$$S_{vN}^{0,0,1} = -(c_2^2 + c_3^2) \ln(c_2^2 + c_3^2) c_1^2 \ln c_1^2 \quad (99)$$

$$= -(1 - \cos^2 \phi \sin^2 \theta) \ln(1 - \cos^2 \phi \sin^2 \theta) - (\cos^2 \phi \sin^2 \theta) \ln(\cos^2 \phi \sin^2 \theta), \quad (100)$$

which quantifies the entanglement between particle *A* and the *BC* subsystem for the state $\psi_{0,0,1}^{ABC}$. This result clearly illustrates that there is no simple one-to-one correspondence between the von Neumann entropy $S_{vN}^{0,0,1}$ and the purity $P_{0,0,1}^A$. While the purity provides a global measure of mixedness, the von Neumann entropy captures the detailed distribution of the eigenvalues of the reduced density matrix. Consequently, $S_{vN}^{0,0,1}$ offers a more refined characterization of how quantum correlations and entanglement are distributed among the accessible eigenstates.

A₂. $n_1 = 0, n_2 \neq 0, n_3 = 0$: The corresponding von Neumann entropy reads

$$S_{vN}^{0,n_2,0} = - \sum_{k=0}^{n_2} \binom{n_2}{k} (b_1^2)^k (b_2^2 + b_3^2)^{n_2-k} \ln \left[\binom{n_2}{k} (b_1^2)^k (b_2^2 + b_3^2)^{n_2-k} \right]. \quad (101)$$

For $n_2 = 1$, it reduces to

$$S_{vN}^{0,1,0} = -(b_2^2 + b_3^2) \ln(b_2^2 + b_3^2) - b_1^2 \ln b_1^2 \quad (102)$$

$$= -\cos^2 \phi \ln(\cos^2 \phi) - \sin^2 \phi \ln(\sin^2 \phi), \quad (103)$$

which is fully consistent with the purity-based measure but additionally highlights the detailed probability weights of each mode.

A₃. $n_1 \neq 0, n_2 = 0, n_3 = 0$: The von Neumann entropy is

$$S_{vN}^{n_1,0,0} = - \sum_{k=0}^{n_1} \binom{n_1}{k} (a_1^2)^k (a_2^2 + a_3^2)^{n_1-k} \ln \left[\binom{n_1}{k} (a_1^2)^k (a_2^2 + a_3^2)^{n_1-k} \right]. \quad (104)$$

For $n_1 = 1$, this simplifies to

$$S_{vN}^{1,0,0} = -(a_2^2 + a_3^2) \ln(a_2^2 + a_3^2) - a_1^2 \ln a_1^2 \quad (105)$$

$$= -(\sin^2 \theta + \cos^2 \theta \sin^2 \phi) \ln(\sin^2 \theta + \cos^2 \theta \sin^2 \phi) - (\cos^2 \theta \cos^2 \phi) \ln(\cos^2 \theta \cos^2 \phi), \quad (106)$$

which quantifies the entanglement entropy for the state $\psi_{1,0,0}^{ABC}$.

Compared to the purity function, the von Neumann entropy provides a richer and more informative description of the quantum correlations in the system. While purity quantifies the overall mixedness of the reduced density matrix, S_{vN} probes the full eigenvalue spectrum, revealing how probability is distributed among the different eigenstates. This enables a finer analysis of the entanglement structure, particularly in non-Gaussian states, where higher-order correlations and mode excitations can generate complex distributions that remain inaccessible to purity alone. As a result, S_{vN} not only distinguishes between separable and entangled states but also highlights the contribution of each mode and the effect of mode hybridization. This makes it especially valuable for studying multipartite continuous-variable systems and for understanding how non-Gaussianity enhances or redistributes entanglement.

4.3. B-AC interaction

For the second configuration, we consider the **B-AC** interaction described by the following Schmidt decomposition given in (65)

$$\psi_{n_1,n_2,n_3}^{ABC}(Zx_1, x_2, x_3) = Z \sum_{l=0}^{n_1+n_2+n_3} \sqrt{\beta_l} \phi_l^B(x_2) \Phi_l^{AC}(x_1, x_3). \quad (107)$$

We subsequently analyze three distinct scenarios: $(n_1 = 0, n_2 = 0, n_3 \neq 0)$, $(n_1 = 0, n_2 \neq 0, n_3 = 0)$, and $(n_1 \neq 0, n_2 = 0, n_3 = 0)$.

B₁. $n_1 = 0, n_2 = 0, n_3 \neq 0$: Substituting $n_1 = 0, n_2 = 0$ into (71), and using (62) together with (50), (53), (56), we get the Schmidt modes

$$\beta_l^{0,0,n_3} = \sum_{k=0}^{n_3-l} \left(A_{0,0,n_3}^{k,l} \right)^2 \quad (108)$$

$$= \frac{n_3! (c_2^2)^l}{l! (n_3-l)!} (c_3^2 + c_1^2)^{n_3-l}, \quad (109)$$

with the condition

$$\text{tr} \rho_{0,0,n_3}^B = \text{tr} \rho_{0,0,n_3}^{AC} = \sum_{l=0}^{n_3} \beta_l^{0,0,n_3} = (c_2^2 + c_1^2 + c_3^2)^{n_3} = 1. \quad (110)$$

We show that the corresponding purity is given by

$$P_{0,0,n_3}^B(\theta, \varphi, \phi) = \left(\frac{n_3!}{l! (n_3-l)!} \right)^2 (c_2^2)^l (c_3^2 + c_1^2)^{n_3-l} = \left((c_1^2 + c_3^2)^2 - c_2^4 \right)^{n_3} \mathcal{P}_{n_3} \left(\frac{(c_1^2 + c_3^2)^2 + c_2^4}{(c_1^2 + c_3^2)^2 - c_2^4} \right) \quad (111)$$

$$= (\cos^2 \phi \sin^2 \theta + (\cos \theta \cos \varphi + \sin \theta \sin \phi \sin \varphi)^2 - \cos^2 \theta \sin^2 \varphi)^{n_3} \quad (112)$$

$$\mathcal{P}_{n_3} \left(\frac{(\cos^2 \phi \sin^2 \theta + (\cos \theta \cos \varphi + \sin \theta \sin \phi \sin \varphi)^2)^2 + (\cos \theta \sin \varphi - \sin \theta \cos \phi \sin \varphi)^4}{\cos^2 \phi \sin^2 \theta + (\cos \theta \cos \varphi + \sin \theta \sin \phi \sin \varphi)^2 - \cos^2 \theta \sin^2 \varphi} \right).$$

B₂. $n_1 = 0, n_2 \neq 0, n_3 = 0$: We analyze (71) under the conditions $n_1 = 0$, and $n_3 = 0$, along with (62), (48), (52), and (55) to obtain the Schmidt modes

$$\beta_k^{0,n_2,0} = \sum_{l=0}^{n_2-k} \left(A_{0,n_2,0}^{k,l} \right)^2 \quad (113)$$

$$= \frac{n_2! \binom{n_2}{k}}{k!(n_2-k)!} (b_1^2 + b_3^2)^{n_2-k}, \quad (114)$$

leading to the condition

$$\text{tr} \rho_{0,n_2,0}^B = \text{tr} \rho_{0,n_2,0}^{AC} = \sum_{k=0}^{n_2} \beta_k^{0,n_2,0} = (b_1^2 + b_3^2 + b_2^2)^{n_2} = 1. \quad (115)$$

The associated purity reads as

$$P_{0,n_2,0}^B(\theta, \varphi, \phi) = \left((b_1^2 + b_3^2)^2 - b_2^4 \right)^{n_2} \mathcal{P}_{n_2} \left(\frac{(b_1^2 + b_3^2)^2 + b_2^4}{(b_1^2 + b_3^2)^2 - b_2^4} \right) \quad (116)$$

$$= (\sin^2 \phi - \cos^2 \varphi \cos^2 \phi + \cos^2 \phi \sin^2 \varphi)^{n_2} \mathcal{P}_{n_2} \left(\frac{(\sin^2 \phi + \cos^2 \phi \sin^2 \varphi)^2 + \cos^4 \phi \cos^4 \varphi}{\sin^2 \phi - \cos^2 \varphi \cos^2 \phi + \cos^2 \phi \sin^2 \varphi} \right). \quad (117)$$

and for $n_2 = 1, 2$, we get

$$P_{0,1,0}^B(\theta, \varphi, \phi) = (\sin^2 \phi + \cos^2 \phi \sin^2 \varphi)^2 + \cos^4 \phi \cos^4 \varphi, \quad (118)$$

$$P_{0,2,0}^A(\theta, \varphi, \phi) = \frac{3}{2} \left((\sin^2 \phi + \cos^2 \phi \sin^2 \varphi)^2 + \cos^4 \phi \cos^4 \varphi \right)^2 - \frac{1}{2} (\sin^2 \phi - \cos^2 \varphi \cos^2 \phi + \cos^2 \phi \sin^2 \varphi)^2. \quad (119)$$

B₃. $n_1 \neq 0, n_2 = 0, n_3 = 0$: From (71) with $n_2 = 0, n_3 = 0$, together with (62), (48), (51), and (54), we end up with the Schmidt modes

$$\beta_k^{n_1,0,0} = \sum_{l=0}^{n_1-k} \left(A_{n_1,0,0}^{k,l} \right)^2 \quad (120)$$

$$= \frac{n_1! \binom{n_1}{k}}{k!(n_1-k)!} (a_1^2 + a_3^2)^{n_1-k}, \quad (121)$$

and the condition is met

$$\text{tr} \rho_{n_1,0,0}^B = \text{tr} \rho_{n_1,0,0}^{AC} = \sum_{k=0}^{n_1} \beta_k^{n_1,0,0} = (a_1^2 + a_3^2 + a_2^2)^{n_1} = 1. \quad (122)$$

As a result, we obtain the purity

$$P_{n_1,0,0}^B(\theta, \varphi, \phi) = \left((a_1^2 + a_3^2)^2 - a_2^4 \right)^{n_1} \mathcal{P}_{n_1} \left(\frac{(a_1^2 + a_3^2)^2 + a_2^4}{(a_1^2 + a_3^2)^2 - a_2^4} \right) \quad (123)$$

$$= ((\cos \theta \cos \phi)^2 + (\cos \theta \sin \phi \sin \varphi - \sin \theta \cos \varphi)^2 - (\sin \theta \sin \varphi + \cos \theta \cos \varphi \sin \phi)^2)^{n_1} \mathcal{P}_{n_1} \left(\frac{((\cos \theta \cos \phi)^2 + (\cos \theta \sin \phi \sin \varphi - \sin \theta \cos \varphi)^2 + (\sin \theta \sin \varphi + \cos \theta \cos \varphi \sin \phi)^4)}{(\cos \theta \cos \phi)^2 + (\cos \theta \sin \phi \sin \varphi - \sin \theta \cos \varphi)^2 - (\sin \theta \sin \varphi + \cos \theta \cos \varphi \sin \phi)^2} \right), \quad (124)$$

which displays the results for $n_1 = 1, 2$

$$P_{1,0,0}^B(\theta, \varphi, \phi) = ((\cos \theta \cos \phi)^2 + (\cos \theta \sin \phi \sin \varphi - \sin \theta \cos \varphi)^2 + (\sin \theta \sin \varphi + \cos \theta \cos \varphi \sin \phi)^4), \quad (125)$$

$$P_{2,0,0}^B(\theta, \varphi, \phi) = \frac{3}{2} \left(((\cos \theta \cos \phi)^2 + (\cos \theta \sin \phi \sin \varphi - \sin \theta \cos \varphi)^2 + (\sin \theta \sin \varphi + \cos \theta \cos \varphi \sin \phi)^4)^2 - \frac{1}{2} ((\cos \theta \cos \phi)^2 + (\cos \theta \sin \phi \sin \varphi - \sin \theta \cos \varphi)^2 - (\sin \theta \sin \varphi + \cos \theta \cos \varphi \sin \phi)^2)^2 \right). \quad (126)$$

4.4. C-AB interaction

To provide a comprehensive overview of all possible configurations, we examine the **C-AB** interaction in the context of the Schmidt decomposition given in (66)

$$\psi_{n_1,n_2,n_3}^{ABC}(Zx_1, x_2, x_3) = \sum_{l=0}^{n_1+n_2+n_3} \sqrt{\beta_l} \phi_l^B(x_2) \Phi_l^{AC}(x_1, x_3). \quad (127)$$

Again, we examine the three different scenarios: $(n_1 = 0, n_2 = 0, n_3 \neq 0)$, $(n_1 = 0, n_2 \neq 0, n_3 = 0)$, and $(n_1 \neq 0, n_2 = 0, n_3 = 0)$.

C₁. $n_1 = 0, n_2 = 0, n_3 \neq 0$: Substituting $n_1 = 0$ and $n_2 = 0$ into (72), together with (62), (50), (53), and (56), we obtain the following Schmidt modes

$$\gamma_\sigma \equiv \gamma_\sigma^{0,0,n_3} = \sum_{k=0}^{n_3-\sigma} \left(A_{0,0,n_3}^{k,n_3-k-\sigma} \right)^2 \quad (128)$$

$$= \frac{n_3! (c_3^2)^\sigma}{\sigma! (n_3 - \sigma)!} (c_2^2 + c_1^2)^{n_3 - \sigma}, \quad (129)$$

verifying the condition

$$\text{tr} \rho_{0,0,n_3}^C = \text{tr} \rho_{0,0,n_3}^{AB} = \sum_{\sigma=0}^{n_3} \gamma_\sigma^{0,0,n_3} = (c_2^2 + c_1^2 + c_3^2)^{n_3} = 1. \quad (130)$$

Then the appropriate purity takes the form

$$P_{0,0,n_3}^C(\theta, \varphi, \phi) = \left((c_1^2 + c_2^2)^2 - c_3^4 \right)^{n_3} \mathcal{P}_{n_3} \left(\frac{(c_1^2 + c_2^2)^2 + c_3^4}{(c_1^2 + c_2^2)^2 - c_3^4} \right) \quad (131)$$

$$= (\cos^2 \phi \sin^2 \theta + (\cos \theta \sin \varphi - \sin \theta \cos \varphi \sin \phi)^2 - (\cos \theta \cos \varphi + \sin \theta \sin \phi \sin \varphi)^2)^{n_3} \mathcal{P}_{n_3} \left(\frac{(\cos^2 \phi \sin^2 \theta + (\cos \theta \sin \varphi - \sin \theta \cos \varphi \sin \phi)^2 + (\cos \theta \cos \varphi + \sin \theta \sin \phi \sin \varphi)^4)}{\cos^2 \phi \sin^2 \theta + (\cos \theta \sin \varphi - \sin \theta \cos \varphi \sin \phi)^2 - (\cos \theta \cos \varphi + \sin \theta \sin \phi \sin \varphi)^2} \right). \quad (132)$$

C₂. $n_1 = 0, n_2 \neq 0, n_3 = 0$: We analyze (72) under the conditions $n_1 = 0$ and $n_3 = 0$, together with (62), (48), (52), and (55). This yields the corresponding Schmidt modes

$$\gamma_\sigma^{0,n_2,0} = \sum_{k=0}^{n_2-\sigma} \left(A_{0,n_2,0}^{k,n_2-k-\sigma} \right)^2 = \frac{n_2! (b_3^2)^\sigma}{\sigma! (n_2 - \sigma)!} (b_1^2 + b_2^2)^{n_2 - \sigma}. \quad (133)$$

Using these quantities, we can verify the standard von Neumann normalization condition

$$\text{tr} \rho_{0,n_2,0}^C = \text{tr} \rho_{0,n_2,0}^{AB} = \sum_{\sigma=0}^{n_2} \gamma_\sigma^{0,n_2,0} = (b_1^2 + b_2^2 + b_3^2)^{n_2} = 1. \quad (134)$$

As a result, we get the purity

$$P_{n_1,n_2,n_3}^C(\theta, \varphi, \phi) = P_{n_1,n_2,n_3}^{AB}(\theta, \varphi, \phi) = \text{tr} \left(\rho_{n_1,n_2,n_3}^C \right)^2 = \sum_{\sigma=0}^{n_1+n_2+n_3} \left(\gamma_\sigma^{n_1,n_2,n_3} \right)^2. \quad (135)$$

In particular for $n_1 = 0, n_2 \neq 0, n_3 = 0$, we obtain the following

$$P_{0,n_2,0}^C(\theta, \varphi, \phi) = \sum_{\sigma=0}^{n_2} \left(\gamma_\sigma^{0,n_2,0} \right)^2 = \left((b_1^2 + b_2^2)^2 - b_3^4 \right)^{n_2} \mathcal{P}_{n_2} \left(\frac{(b_1^2 + b_2^2)^2 + b_3^4}{(b_1^2 + b_2^2)^2 - b_3^4} \right) \quad (136)$$

$$= (\sin^2 \phi + \cos^2 \phi \cos^2 \varphi - \cos^2 \phi \sin^2 \varphi)^{n_2} \mathcal{P}_{n_2} \left(\frac{(\sin^2 \phi + \cos^2 \phi \cos^2 \varphi)^2 + \cos^4 \phi \sin^4 \varphi}{\sin^2 \phi + \cos^2 \phi \cos^2 \varphi - \cos^2 \phi \sin^2 \varphi} \right). \quad (137)$$

For further illustration, such that $n_2 = 1, 2$, we find

$$P_{0,1,0}^C(\theta, \varphi, \phi) = (\sin^2 \phi + \cos^2 \phi \cos^2 \varphi)^2 + \cos^4 \phi \sin^4 \varphi, \quad (138)$$

$$P_{0,2,0}^C(\theta, \varphi, \phi) = \frac{3}{2} \left((\sin^2 \phi + \cos^2 \phi \cos^2 \varphi)^2 + \cos^4 \phi \sin^4 \varphi \right)^2 - \frac{1}{2} (\sin^2 \phi + \cos^2 \phi \cos^2 \varphi - \cos^2 \phi \sin^2 \varphi)^2. \quad (139)$$

C₃. $n_1 \neq 0, n_2 = 0, n_3 = 0$: In the last scenario, starting from (71) with $n_2 = 0$ and $n_3 = 0$, together with (62), (48), (51), and (54), we arrive at the Schmidt modes

$$\gamma_\sigma^{n_1,0,0} = \sum_{l=0}^{n_1-\sigma} \left(A_{n_1,0,0}^{l,n_1-l-\sigma} \right)^2 \quad (140)$$

$$= \frac{n_1! (a_3^2)^\sigma}{\sigma! (n_1 - \sigma)!} (a_1^2 + a_2^2)^{n_1 - \sigma}, \quad (141)$$

and we have the condition

$$\text{tr} \rho_{n_1,0,0}^C = \text{tr} \rho_{n_1,0,0}^{AB} = \sum_{\sigma=0}^{n_1} \gamma_\sigma^{n_1,0,0} = (a_1^2 + a_2^2 + a_3^2)^{n_1} = 1. \quad (142)$$

We show that the corresponding purity takes the form

$$P_{n_1,0,0}^C(\theta, \varphi, \phi) = \left((a_1^2 + a_2^2)^2 - a_3^4 \right)^{n_1} \mathcal{P}_{n_1} \left(\frac{(a_1^2 + a_2^2)^2 + a_3^4}{(a_1^2 + a_2^2)^2 - a_3^4} \right) \quad (143)$$

$$= (\cos^2 \theta \cos^2 \phi + (\sin \theta \sin \varphi + \cos \theta \cos \varphi \sin \phi)^2 - (\cos \theta \sin \phi \sin \varphi - \sin \theta \cos \varphi)^2)^{n_1}$$

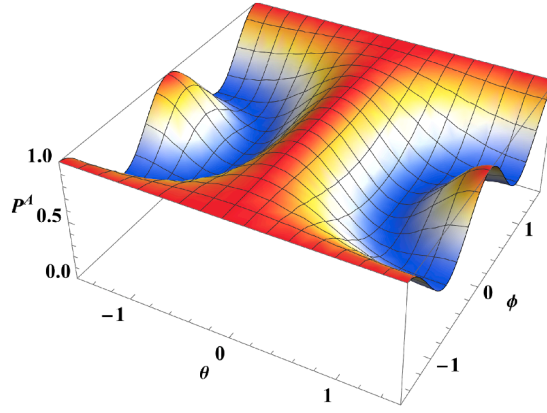


Fig. 1. (Color online) The purity function $P_{0,0,1}^A = P^A$ versus the two angles θ and $\phi \in [-\pi, \pi]$.

$$P_{n_1} \left(\frac{(\cos^2 \theta \cos^2 \phi + (\sin \theta \sin \varphi + \cos \theta \cos \varphi \sin \phi)^2 + (\cos \theta \sin \phi \sin \varphi - \sin \theta \cos \varphi)^4)}{\cos^2 \theta \cos^2 \phi + (\sin \theta \sin \varphi + \cos \theta \cos \varphi \sin \phi)^2 - (\cos \theta \sin \phi \sin \varphi - \sin \theta \cos \varphi)^2} \right), \quad (144)$$

which results in the following for $n_1 = 1, 2$

$$P_{1,0,0}^C(\theta, \varphi, \phi) = (\cos^2 \theta \cos^2 \phi + (\sin \theta \sin \varphi + \cos \theta \cos \varphi \sin \phi)^2 + (\cos \theta \sin \phi \sin \varphi - \sin \theta \cos \varphi)^4), \quad (145)$$

$$P_{2,0,0}^C(\theta, \varphi, \phi) = \frac{3}{2} \left((\cos^2 \theta \cos^2 \phi + (\sin \theta \sin \varphi + \cos \theta \cos \varphi \sin \phi)^2 + (\cos \theta \sin \phi \sin \varphi - \sin \theta \cos \varphi)^4) \right)^2 - \frac{1}{2} (\cos^2 \theta \cos^2 \phi + (\sin \theta \sin \varphi + \cos \theta \cos \varphi \sin \phi)^2 - (\cos \theta \sin \phi \sin \varphi - \sin \theta \cos \varphi)^2)^2. \quad (146)$$

5. Examples and numerical analysis

To highlight the main differences between the various interactions and to provide a clearer understanding of the cases considered, we focus on (81), (111), and (131). These correspond to the configuration in which the quantum numbers satisfy $n_1 = 0$, $n_2 = 0$, and $n_3 \neq 0$. For further analysis, we consider the cases $n_3 = 1$ and $n_3 = 2$.

5.1. Purity function $P_{0,0,n_3}^A$

As far as (81) is concerned, for $n_3 = 1, 2$, we get

$$P_{0,0,1}^A(\theta, \varphi, \phi) = (\cos^2 \theta + \sin^2 \theta \sin^2 \phi)^2 + \sin^4 \theta \cos^4 \phi, \quad (147)$$

$$P_{0,0,2}^A(\theta, \varphi, \phi) = \frac{3}{2} \left((\cos^2 \theta + \sin^2 \theta \sin^2 \phi)^2 + \sin^4 \theta \cos^4 \phi \right)^2 - \frac{1}{2} \left((\cos^2 \theta + \sin^2 \theta \sin^2 \phi)^2 - \sin^4 \theta \cos^4 \phi \right)^2. \quad (148)$$

In Fig. 1, we plot the purity function $P_{0,0,1}^A$ (147) as a function of the two angles θ and ϕ . We observe several peaks and valleys, indicating significant variation in purity with respect to these angles. The peaks correspond to higher purity values, approaching 1, while the valleys represent lower purity values, around 0.5. These variations suggest that certain angular configurations lead to a more pure (less mixed) quantum state, whereas others lead to a more mixed state. This sensitivity of purity to changes in the angles θ and ϕ may reflect how the quantum state evolves under transformations parameterized by these angles. The surface also exhibits periodicity, a common feature of functions depending on angular variables, which may be related to symmetries in the underlying quantum system, such as rotational symmetry. The pattern of peaks and troughs likely reflects the intrinsic symmetries of the quantum state, possibly related to rotational invariance or to the specific symmetries of the Hamiltonian governing the system.

5.2. Purity function $P_{0,0,n_3}^B$

Regarding (111) with $n_3 = 1, 2$, we find

$$P_{0,0,1}^B(\theta, \varphi, \phi) = (\cos^2 \phi \sin^2 \theta + (\cos \theta \cos \varphi + \sin \theta \sin \phi \sin \varphi)^2 + (\cos \theta \sin \varphi - \sin \theta \cos \varphi \sin \phi)^4), \quad (149)$$

$$P_{0,0,2}^B(\theta, \varphi, \phi) = \frac{3}{2} \left((\cos^2 \phi \sin^2 \theta + (\cos \theta \cos \varphi + \sin \theta \sin \phi \sin \varphi)^2 + (\cos \theta \sin \varphi - \sin \theta \cos \varphi \sin \phi)^4) \right)^2 - \frac{1}{2} (\cos^2 \phi \sin^2 \theta + (\cos \theta \cos \varphi + \sin \theta \sin \phi \sin \varphi)^2 - \cos^2 \theta \sin^2 \varphi)^2. \quad (150)$$

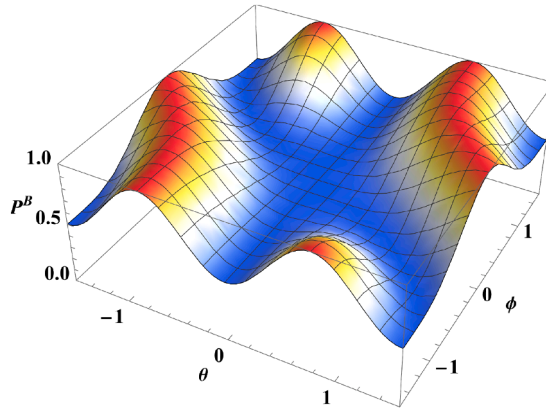


Fig. 2. (Color online) The purity function $P^B_{0,0,1} = P^B$ versus the two angles θ and $\phi \in [-\pi, \pi]$ with $\varphi = \frac{\pi}{2}$.

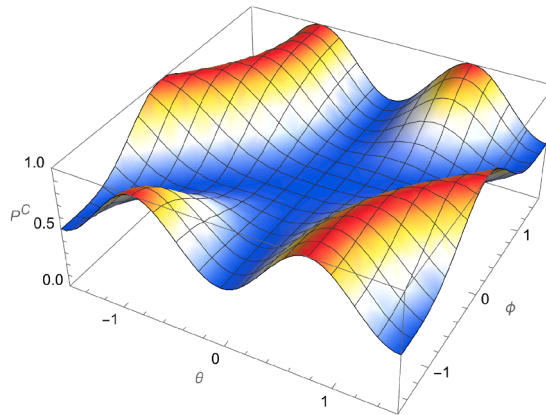


Fig. 3. (Color online) The purity function $P^C_{0,0,1} = P^C$ versus the two angles θ and $\phi \in [-\pi, \pi]$ with $\varphi = \frac{\pi}{4}$.

Fig. 2 shows the purity $P^B_{0,0,1}$ (149) as a function of the two angles θ and $\phi \in [-\pi, \pi]$ with $\varphi = \frac{\pi}{4}$. When this purity remains consistently lower than in Fig. 1, it may indicate that the entanglement dynamics are less favorable for this particular interaction. We note that any oscillations in purity may reflect coherent oscillations between the oscillators, corresponding to a periodic exchange of entanglement. In addition, a notable increase in purity at certain parameter values may signal resonant interactions or optimal conditions for entanglement generation.

5.3. Purity function $P^C_{0,0,n_3}$

By choosing $n_3 = 1, 2$, (131) gives

$$P^C_{0,0,1}(\theta, \varphi, \phi) = ((\cos \theta \sin \varphi - \sin \theta \cos \varphi \sin \phi)^2 + \cos^2 \phi \sin^2 \theta)^2 + (\cos \theta \cos \varphi + \sin \theta \sin \phi \sin \varphi)^4, \quad (151)$$

$$P^C_{0,0,2}(\theta, \varphi, \phi) = \frac{3}{2} \left(((\cos \theta \sin \varphi - \sin \theta \cos \varphi \sin \phi)^2 + \cos^2 \phi \sin^2 \theta)^2 + (\cos \theta \cos \varphi + \sin \theta \sin \phi \sin \varphi)^4 \right)^2 - \frac{1}{2} ((\cos \theta \sin \varphi - \sin \theta \cos \varphi \sin \phi)^2 + \cos^2 \phi \sin^2 \theta - (\cos \theta \cos \varphi + \sin \theta \sin \phi \sin \varphi)^2)^2. \quad (152)$$

Fig. 3 shows the purity $P^C_{0,0,1}$ (151) as a function of the two angles θ and $\phi \in [-\pi, \pi]$ with $\varphi = \frac{\pi}{4}$. If this interaction exhibits the highest purity of the three cases, it indicates that **C-AB** configuration is particularly well entangled compared with the **A-BC** and **B-AC** configurations, suggesting an effective coupling within the system. A gradual increase or a consistently high purity may indicate stable entanglement over a range of parameters, highlighting the robustness of this interaction. From a physical perspective, these results suggest that the *C* subsystem plays a crucial role in maintaining or enhancing entanglement within the system, which could be advantageous for quantum information applications.

By comparing the purities and the corresponding numerical plots, we can evaluate the effectiveness of different interactions in generating entanglement. Some interactions may be more favorable under certain conditions, exhibiting higher purity and greater stability, whereas others may display reduced purity or weaker correlations. These results provide insights into the dynamic interplay between the subsystems and show how different coupling configurations can lead to distinct entanglement behavior. They highlight

the important role played by the interaction structure in shaping entanglement properties and suggest possible strategies for optimizing these interactions. In particular, by appropriately selecting the coupling parameters and configurations, it may be possible to enhance entanglement in practical quantum systems. Such optimization could be beneficial for applications in quantum information processing, quantum communication, and quantum computation. More broadly, these findings may serve as a useful starting point for future theoretical and experimental investigations aimed at maximizing entanglement generation through carefully engineered interaction schemes, thereby contributing to the development of more efficient quantum technologies.

6. Non-Gaussian characteristics

6.1. Ground-state versus first-excited-state properties

For the Gaussian ground state $\psi_{0,0,0}^{ABC}$

$$\psi_{0,0,0}^{ABC}(x_1, x_2, x_3) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{3}{4}} e^{-\frac{m\omega}{2\hbar}(\mu_1^2 x_1^2 + \mu_2^2 x_2^2 + \mu_3^2 x_3^2)}, \quad (153)$$

all entanglement measures vanish because the state is separable, i.e., the von Neumann entropy is

$$S_{vN}^{0,0,0} = 0. \quad (154)$$

The first excited state $\psi_{1,0,0}^{ABC}$ takes the form

$$\psi_{1,0,0}^{ABC}(x_1, x_2, x_3) = \frac{1}{\sqrt{2}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{3}{4}} e^{-\frac{m\omega}{2\hbar}(\mu_1^2 x_1^2 + \mu_2^2 x_2^2 + \mu_3^2 x_3^2)} H_1\left(\sqrt{\frac{m\omega}{\hbar}} q_1\right), \quad (155)$$

where the variable is given by

$$q_1 = \mu_1 \cos \theta \cos \phi x_1 - \mu_2 (\sin \theta \sin \phi + \cos \theta \cos \phi \sin \phi) x_2 - \mu_3 (\sin \theta \cos \phi - \cos \theta \sin \phi \sin \phi) x_3. \quad (156)$$

This state is non-Gaussian because it contains the Hermite polynomial $H_1\left(\sqrt{\frac{m\omega}{\hbar}} q_1\right)$, which explicitly reads

$$H_1\left(\sqrt{\frac{m\omega}{\hbar}} q_1\right) = 2\sqrt{\frac{m\omega}{\hbar}} q_1. \quad (157)$$

We aim to compute the reduced density matrix for particle **A**, $\rho_{1,0,0}^A(x_1, x'_1)$, defined as

$$\rho_{1,0,0}^A(x_1, x'_1) = \int \int \psi_{1,0,0}^{ABC}(x_1, x_2, x_3) \psi_{1,0,0}^{*ABC}(x'_1, x_2, x_3) dx_2 dx_3. \quad (158)$$

After performing the Gaussian integrals over x_2 and x_3 , the reduced density matrix becomes

$$\rho_{1,0,0}^A(x_1, x'_1) = \sqrt{\frac{m\omega}{\pi\hbar}} \mu_1 \left(\frac{2m\omega\mu_1^2}{\hbar} \cos^2 \theta \cos^2 \phi x_1 x'_1 + \sin^2 \theta + \cos^2 \theta \sin^2 \phi \right) e^{-\frac{m\omega}{2\hbar} \mu_1^2 (x_1^2 + x'^2_1)}, \quad (159)$$

which possesses at most the following two nonzero eigenvalues

$$\lambda_0 = \cos^2 \theta \cos^2 \phi, \quad \lambda_1 = \sin^2 \theta + \cos^2 \theta \sin^2 \phi. \quad (160)$$

The von Neumann entropy of the reduced density matrix $\rho_{1,0,0}^A$ is given by the binary entropy function

$$S_{vN}^{1,0,0} = -(\cos^2 \theta \cos^2 \phi) \ln(\cos^2 \theta \cos^2 \phi) - (\sin^2 \theta + \cos^2 \theta \sin^2 \phi) \ln(\sin^2 \theta + \cos^2 \theta \sin^2 \phi), \quad (161)$$

which depends solely on the parameter $u = \cos^2 \theta \cos^2 \phi$, with $0 \leq u \leq 1$. The entropy reaches its maximum value $\ln 2$ at $u = \frac{1}{2}$ and vanishes at the extremal points $u = 0$ and $u = 1$. Consequently, one finds the following bound

$$S_{vN}^{0,0,0} \leq S_{vN}^{1,0,0} \leq \ln 2. \quad (162)$$

These results indicate that, in contrast to the Gaussian ground state $\psi_{0,0,0}^{ABC}(x_1, x_2, x_3)$, which is separable and unentangled, the excited non-Gaussian state $\psi_{1,0,0}^{ABC}(x_1, x_2, x_3)$ can generate finite bipartite entanglement. Although the precise amount of entanglement depends on the mixing parameters, the presence of non-Gaussianity generally modifies the entanglement structure, and can enhance quantum correlations beyond those achievable in purely Gaussian states.

6.2. Demonstration of non-Gaussianity

We can demonstrate the emergence of non-Gaussianity by employing two complementary approaches. The first is moment analysis, which probes higher-order statistical moments of the reduced quadrature distributions. For the Gaussian ground state, all moments are fully determined by the second-order moments, and higher-order cumulants vanish. In contrast, for the first excited state, deviations in the higher-order moments signal a departure from Gaussian statistics, directly reflecting the presence of non-Gaussian features induced by the excitation.

The second approach is based on the negativity of the Wigner function, which provides a phase-space characterization of the quantum state. While the Wigner function associated with the Gaussian ground state remains strictly positive over the entire phase

space, the Wigner function of the first excited state develops negative regions. This negativity constitutes a clear signature of non-Gaussianity and nonclassicality. Moreover, these non-Gaussian features are closely connected to the modifications of the entanglement structure, as reflected by the increase in von Neumann entropy in the excited state compared to the ground state.

Moment analysis: For $\psi_{1,0,0}^{ABC}$ the probability density for particle (tracing out **B** and **C**) is

$$p_{1,0,0}^A(x_1) = \sqrt{\frac{m\omega}{\pi\hbar}} \mu_1 \left(\frac{2m\omega\mu_1^2}{\hbar} \cos^2\theta \cos^2\phi x_1^2 + \sin^2\theta + \cos^2\theta \sin^2\phi \right) e^{-\frac{m\omega}{\hbar} \mu_1^2 x_1^2}. \quad (163)$$

The second and fourth moments are

$$\langle x_1^2 \rangle_{1,0,0} = \frac{\hbar(1 + 2\cos^2\theta \cos^2\phi)}{2m\omega\mu_1}, \quad \langle x_1^4 \rangle_{1,0,0} = \frac{3\hbar^2(1 + 4\cos^2\theta \cos^2\phi)}{(2m\omega\mu_1)^2}, \quad (164)$$

giving the kurtosis

$$\kappa = \frac{\langle x_1^4 \rangle_{1,0,0}}{\langle x_1^2 \rangle_{1,0,0}^2} = \frac{3(1 + 4\cos^2\theta \cos^2\phi)}{(1 + 2\cos^2\theta \cos^2\phi)^2}. \quad (165)$$

For a Gaussian distribution, the kurtosis is $\kappa = \frac{\langle x_1^4 \rangle_{0,0,0}}{\langle x_1^2 \rangle_{0,0,0}^2} = 3$. Any deviation from this value provides a clear indication of non-Gaussianity.

Wigner function negativity: We now compute the Wigner function for the full three-mode state. For a multidimensional wavefunction, the Wigner function is defined as

$$W_{1,0,0}^{ABC}(x_1, x_2, x_3, p_1, p_2, p_3) = \frac{1}{(\pi\hbar)^3} \int \psi_{1,0,0}^{ABC}(x_1 + y_1, x_2 + y_2, x_3 + y_3) \psi_{1,0,0}^{*ABC}(x_1 - y_1, x_2 - y_2, x_3 - y_3) \\ \times e^{-\frac{2i}{\hbar}(p_1 y_1 + p_2 y_2 + p_3 y_3)} dy_1 dy_2 dy_3. \quad (166)$$

By performing the integration, we get

$$W_{1,0,0}^{ABC}(x_1, x_2, x_3, p_1, p_2, p_3) = \frac{1}{(\pi\hbar)^3} e^{-\frac{m\omega}{\hbar}(\mu_1^2 x_1^2 + \mu_2^2 x_2^2 + \mu_3^2 x_3^2) - \frac{1}{m\omega\hbar} \left(\frac{p_1^2}{\mu_1^2} + \frac{p_2^2}{\mu_2^2} + \frac{p_3^2}{\mu_3^2} \right)} \left(\frac{2m\omega}{\hbar} q_1^2 + \frac{2}{m\omega\hbar} (q_1^{(p)})^2 - 1 \right), \quad (167)$$

where the variable $q_1^{(p)}$ is given by

$$q_1^{(p)} = \frac{1}{\mu_1} \cos\theta \cos\phi p_1 - \frac{1}{\mu_2} (\sin\theta \sin\phi + \cos\theta \cos\phi \sin\phi) p_2 - \frac{1}{\mu_3} (\sin\theta \cos\phi - \cos\theta \sin\phi \sin\phi) p_3. \quad (168)$$

This Wigner function is not everywhere positive, reflecting the non-Gaussian nature of the excited state. We now consider the reduced Wigner function for the subsystem **A**, obtained by integrating out the modes **B** and **C**. That is

$$W_{1,0,0}^A(x_1, p_1) = \int W_{1,0,0}^{ABC}(x_1, x_2, x_3, p_1, p_2, p_3) dx_2 dx_3 dp_2 dp_3. \quad (169)$$

We can split the integral into two parts: one involving q_1^2 , which depends only on the x variables, and another involving $(q_1^{(p)})^2$, which depends only on the p variables, together with a constant term. The reduced Wigner function for the subsystem **A** is

$$W_{1,0,0}^A(x_1, p_1) = \frac{1}{\pi\hbar} e^{-\frac{m\omega}{\hbar} \mu_1^2 x_1^2 - \frac{1}{m\omega\hbar} \frac{p_1^2}{\mu_1^2}} \left(\frac{2m\omega\mu_1^2 \cos^2\theta \cos^2\phi}{\hbar} x_1^2 + \frac{2\cos^2\theta \cos^2\phi}{\mu_1^2 m\omega\hbar} p_1^2 + 1 - 2\cos^2\theta \cos^2\phi \right), \quad (170)$$

which can take negative values, providing a clear signature of non-Gaussianity. In contrast, for the Gaussian ground state, the Wigner function $W_{0,0,0}^A$ is

$$W_{0,0,0}^A(x_1, p_1) = \frac{1}{\pi\hbar} e^{-\frac{m\omega}{\hbar} \mu_1^2 x_1^2 - \frac{1}{m\omega\hbar} \frac{p_1^2}{\mu_1^2}}, \quad (171)$$

which is everywhere positive.

7. Conclusion

We developed a method to analytically determine the energy eigenstates of three coupled quantum harmonic oscillators, expressing the tripartite wavefunction in a separable form. By applying the Schmidt decomposition, the wavefunction was represented as a sum of orthogonal product states describing the bipartitions. Integrating over the Schmidt modes allowed us to obtain closed-form expressions for the Schmidt coefficients that quantify the bipartite entanglement. In particular, we showed that these coefficients can be expressed in terms of a hypergeometric function, providing a compact characterization of the reduced density matrices. By tuning the interaction strengths, we examined how the Schmidt coefficients evolve and how entanglement is distributed across the different

bipartitions. As the couplings vary, this analysis provides insight into the multipartite entanglement structure within a system of three coupled harmonic oscillators. Overall, this work establishes a framework for analyzing static bipartite entanglement in this model.

In summary, we have developed an analytical framework for the characterization of tripartite entanglement, providing deeper insights into the entanglement structure of quantum systems with three coupled subsystems. These results not only enhance our understanding of multipartite entanglement, but also have broader implications for a variety of quantum systems, including quantum information processing, quantum computing, and entanglement-based technologies. Our approach opens the possibility for more precise control and manipulation of entangled states, which is critical for advancing both theoretical and practical applications in quantum science.

Finally, we emphasize that in this work we have applied the approach introduced by Makarov [16] for the computation of Schmidt numbers using Schmidt decomposition to a non-Gaussian bipartite system. This approach provides valuable insights into the entanglement properties of such systems. Building on this foundation, we extended the methodology to a non-Gaussian tripartite system, which presents additional complexities and opportunities for exploration. In our study, we focused on a specific class of tripartite systems in order to provide a detailed analysis of their structure and behavior. This focused approach allowed us to highlight key aspects of entanglement and information distribution in these systems. We note that a systematic classification of the different tripartite configurations and their associated Schmidt numbers constitutes a substantial task. Therefore, we have reserved a comprehensive classification analysis for future work in which we aim to systematically explore the broader implications of our findings for different classes of non-Gaussian tripartite systems. This future work will build on our results and contribute to a more complete understanding of the entanglement structures that arise in multipartite non-Gaussian systems.

We emphasize that the quadratic Hamiltonian (1) can be realized in several established experimental platforms, such as trapped ions [47], superconducting circuits [40], and optical lattices [48]. These systems allow precise control over bosonic modes and enable the implementation of engineered couplings. The ground state of such a Hamiltonian is a separable Gaussian state, which can typically be prepared using standard ground-state cooling techniques [49]. However, our focus lies on excited states, which are non-Gaussian and, therefore, present additional challenges in their preparation. Accessing specific excited states, such as Fock states labeled by the excitation number n , has nevertheless been demonstrated experimentally in multiple platforms. In trapped-ion systems, sideband transitions have enabled the preparation of individual motional Fock states, and their superpositions [50]. In superconducting circuits, Fock states have been engineered in resonators coupled to tunable qubits using optimal control techniques [51]. Conditional measurements and projective operations have been used to generate non-Gaussian states in both photonic and superconducting systems [52]. Furthermore, reservoir engineering and adiabatic passage methods have been proposed and implemented to stabilize or selectively reach nontrivial eigenstates of Gaussian systems [53].

Although our analysis focused on a particular three-mode quadratic Hamiltonian, the methodology developed here is not limited to this case. In fact, the approach can be naturally extended to more general classes of quadratic Hamiltonians, especially those that can be diagonalized via canonical transformations into a set of decoupled harmonic modes. For larger systems, such as N -mode bosonic systems with arbitrary pairwise couplings, constructing excited states in the normal-mode basis remains a valid procedure, and analyzing the entanglement structure through bipartite Schmidt decompositions is still possible. Systems with symmetric topologies (e.g., star or chain geometries) may even admit partially analytical treatment due to mode degeneracies or conserved quantities. However, as the system size increases, the Fock-space representation becomes more complex, and the non-Gaussian features of excited states become increasingly intricate, making a full analytical treatment progressively more challenging. In such cases, numerical tools are essential for complementing the analytical framework. Nevertheless, the core structure of our method, which relies on the solvability of quadratic models and the construction of non-Gaussian excitations, remains robust and adaptable to a broad class of Hamiltonians.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Particular cases

To illustrate the consistency of our formalism with known results in the literature, let us examine several limiting cases characterized by different values of the coupling parameters. In particular, to recover the solutions for two coupled harmonic oscillators in the variables (x_1, x_2) , it is sufficient to consider the limit $D_{13}, D_{23} \rightarrow 0$, which corresponds to $J_{13}, J_{23} \rightarrow 0$. Under these conditions, the Hamiltonian (1) reduces to the form

$$H = H_0 + \frac{p_3^2}{2m_3} + \frac{1}{2}m_3\omega_3^2x_3^2, \quad (\text{A.1})$$

where H_0 is the Hamiltonian of two coupled harmonic oscillators in the variables (x_1, x_2) , defined as

$$H_0 = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{1}{2}m_1\omega_1^2x_1^2 + \frac{1}{2}m_2\omega_2^2x_2^2 + \frac{1}{2}D_{12}x_1x_2. \quad (\text{A.2})$$

If we impose the conditions $\tilde{J}_{13} = 0$ and $\tilde{J}_{23} = 0$ in (31–33), we find that $\theta \rightarrow 0$ and $\varphi \rightarrow 0$. Consequently, (31) reduces to the relation

$$\frac{2J_{12}}{\omega_1^2 - \omega_2^2} \rightarrow \frac{2\tilde{J}_{12}}{\tilde{\omega}_1^2 - \tilde{\omega}_2^2} = -\tan 2\phi. \quad (\text{A.3})$$

It is worth noting that these conditions coincide with those employed in our previous study [54] to decouple the system of two harmonic oscillators in the variables (x_1, x_2) . Then, the Hamiltonian is transformed into

$$H = \left(\frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{m}{2}kq_1^2 + \frac{m}{2}kq_2^2 \right) + \frac{p_3^2}{2m_3} + \frac{1}{2}m\omega_3^2q_3^2, \quad (\text{A.4})$$

with the variables

$$q_1 = \mu_1 \cos \phi x_1 - \mu_2 \sin \phi x_2, \quad q_2 = \mu_1 \sin \phi x_1 + \mu_2 \cos \phi x_2, \quad q_3 = \mu_3 x_3. \quad (\text{A.5})$$

In this case, the parameters (48–54) become

$$a_1 = \cos \phi, \quad b_1 = \sin \phi, \quad c_1 = 0, \quad (\text{A.6})$$

$$a_2 = -\sin \phi, \quad b_2 = \cos \phi, \quad c_2 = 0, \quad (\text{A.7})$$

$$a_3 = 0, \quad b_3 = 0, \quad c_3 = 1, \quad (\text{A.8})$$

which lead to the following four conditions that are fulfilled by the quantum numbers

$$i_2 = k - i_1, \quad j_1 = n_1 - i_1, \quad j_2 = n_2 - k + i_1, \quad l = n_1 + n_2 - k. \quad (\text{A.9})$$

These can be substituted into (57) to get the Schmidt modes

$$A_{n_1, n_2, n_3}^{k, l} = A_{n_1, n_2}^k \delta_{l, n_1 + n_2 - k}, \quad (\text{A.10})$$

where A_{n_1, n_2}^k is

$$A_{n_1, n_2}^k = \frac{\sqrt{k!(n_1 + n_2 - k)!(-\sin \phi)^{n_1 - k}(\cos \phi)^{n_2 - k}P_k^{(n_1 - k, n_2 - k)}(\cos 2\phi)}}{\sqrt{n_1!n_2!}}. \quad (\text{A.11})$$

Alternatively, to establish a connection between our results and those of Makarov [16], it suffices to apply the two relations [55]

$$P_n^{(\rho, m-n)}(z) = \frac{m!}{n!} \frac{\Gamma(n+\rho+1)}{\Gamma(m+\rho+1)} \left(\frac{z+1}{2} \right)^{n-m} P_m^{(\rho, n-m)}(z), \quad (\text{A.12})$$

$$P_n^{(\rho, \sigma)}(z) = \left(\frac{1-z}{2} \right)^n P_n^{(-\rho-\sigma-2n-1, \sigma)} \left(\frac{z+3}{z-1} \right). \quad (\text{A.13})$$

As a result, we show that

$$\lambda_k = \sum_l \left(A_{n_1, n_2, n_3}^{k, l} \right)^2 = \left(A_{n_1, n_2}^k \right)^2 \quad (\text{A.14})$$

$$= \frac{n_1!n_2! \sin^{2(n_1+k)}(\phi) \cos^{2(n_2-k)}(\phi) \left(P_{n_1}^{(-n_1-n_2-1, n_2-k)} \left(-\frac{2+\tan^2(\phi)}{\tan^2(\phi)} \right) \right)^2}{k!(n_1+n_2-k)!}. \quad (\text{A.15})$$

Now, the state (64) becomes

$$\sum_l A_{n_1, n_2, n_3}^{k, l} \varphi_k(x_1) \phi_l(x_2) \chi_{n_1+n_2+n_3-k-l}(x_3) \rightarrow A_{n_1, n_2}^k \varphi_k(x_1) \phi_{n_1+n_2-k}(x_2) \chi_{n_3}(x_3), \quad (\text{A.16})$$

and the decomposition is reduced to

$$\psi^{ABC} = \sum_k A_{n_1, n_2}^k \varphi_k(x_1) \phi_{n_1+n_2-k}(x_2) \chi_{n_3}(x_3) = \psi^{AB}(x_1, x_2) \chi_{n_3}(x_3), \quad (\text{A.17})$$

where $\psi^{AB}(x_1, x_2)$ is

$$\psi^{AB}(x_1, x_2) = \sum_k A_{n_1, n_2}^k \varphi_k(x_1) \phi_{n_1+n_2-k}(x_2), \quad (\text{A.18})$$

which exactly coincides with the one obtained by Makarov [16].

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