

Research paper

Multistable chaotic dynamics and basin fragmentation in a minimal model for driven active matter

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ABSTRACT

We present a comprehensive numerical analysis of a minimal Lorenz-type model for active matter, derived from hydrodynamic equations and exhibiting complex nonlinear dynamics. The active contribution is retained through the nonlinear coupling term (YZ), while the flow remains strictly dissipative with phase-space contraction rate $-(\sigma + 3)$. Using Lyapunov exponents, isospike diagrams, complexity measures, and time-averaged observables, we explore the global organization of the parameter space and identify rich transitions between periodicity and chaos. We report the emergence of quintuplet points in this class of models, singular structures, rarely discussed in minimal descriptions of active matter, that organize spike-adding and spike-doubling bifurcations and shape local bifurcation cascades. Building on these local structures, we reveal the coexistence of multiple attractors, ranging from fixed points to chaotic states, as well as fragmented basins of attraction that highlight the system's high sensitivity to initial conditions. We further quantify this fragmentation using Basin Entropy analysis. Our results provide new tools for investigating complexity and control in low-dimensional active matter systems.

1. Introduction

Active matter comprises systems whose constituents continuously consume energy and generate motion and stresses far from equilibrium. In wet active matter, the concentration field of active particles couples to the solvent flow, so even minimal stress imbalances can reorganize transport, destabilize homogeneous states, and drive complex temporal responses [1–4]. Reduced models are therefore valuable because they compress this nonequilibrium feedback into a tractable phase-space description while preserving the core competition between forcing, dissipation, and nonlinear saturation [5,6]. Additionally, the study of extended chaotic behaviors and synchronization phenomena has recently advanced analytical tools in diverse non-ideal systems [7–10], providing complementary perspectives for exploring driven active matter.

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A particularly transparent route is provided by active Model H, a scalar hydrodynamic description of active particles suspended in an incompressible, momentum-conserving solvent [6]. In its minimal form, the model couples a conserved concentration field to the flow through active stresses and has been used to rationalize pattern formation and instabilities in wet active media [11,12]. Projecting the leading hydrodynamic modes onto a Galerkin basis yields a modified Lorenz-type system that retains the nonlinear feedback between velocity and concentration while drastically reducing the dimensionality of the original field theory [13,14]. Similar low-dimensional reductions focusing on complex bi-parametric dynamics have proven crucial in diverse hydrodynamic contexts, such as acoustic-gravity waves, thermo-viscoelastic fluids [15–18], and generalized non-autonomous dynamical models [19]. Compared with the classical Lorenz equations for thermal convection, the present system preserves the same three-dimensional dissipative backbone, the $\mathbb{Z}_2$ symmetry $(X, Y, Z) \mapsto (-X, -Y, Z)$, and the quadratic saturation terms involving XY and XZ . It differs, however, in three essential aspects: Y and Z represent concentration modes of an active fluid rather than temperature modes of a passive convective layer, the control parameter r enters the forcing of the X equation, and the model contains the additional quadratic coupling rYZ , which is absent in the classical Lorenz system. Moreover, the geometric coefficient is fixed by the hydrodynamic projection to $b = 2$, instead of the classical value $b = 8/3$ [20]. The present equations should therefore be regarded not as a simple reparametrization of the Lorenz model, but as a minimal active extension with a modified bifurcation geometry.

The main objective of this work is to characterize how this active extension organizes periodicity, chaos, local bifurcation structures, and multistability in the bi-parameter plane. To this end, we combine Lyapunov exponents, isospoke diagrams, LMC complexity, and basin-based diagnostics to resolve both the global control-space organization and the local mechanisms underlying transitions between dynamical states [21–25]. Particular attention is given to quintuplet points, where several periodicity tongues collide, and to the fragmentation of basins of attraction under fixed control parameters.

The paper is organized as follows. Section 2 introduces the low-dimensional model derived from the hydrodynamic equations and the numerical methods employed. Section 3 presents the global exploration of the control plane based on Lyapunov exponents, isospoke diagrams, and complexity metrics. Section 4 focuses on the local organizing structures, with special attention to the quintuplet points, while Section 5 quantifies multistability and basin fragmentation. Final remarks are summarized in Section 6. In addition, in Appendix A is provided a derivation of the recurrence rules governing quintuplet points and verifies their universality through high-resolution maps in distinct regions of the bi-parameter plane.

2. Model and numerical methods

2.1. Theoretical model: modified Lorenz system

Here, we start with a simplified two-dimensional spatial model for wet active matter, which excludes explicitly orientational degrees of freedom. This model allows for a more tractable analysis while still encapsulating the essential dynamics. This model is an effective fluid version of Model H in the Halperin-Hohenberg classification scheme [6], which consider two fields: concentration field $\phi(\mathbf{r}, t)$ of the active particles and the velocity of the fluid medium, $\mathbf{u}(\mathbf{r}, t)$. Recently, the Galerkin approximation has been employed to analyze the system, a method that reduces the complexity of the hydrodynamic equations by projecting them onto a finite set of basis functions [20,26,27]. This approach retains the minimal nontrivial dynamics necessary for capturing the essential dynamics of the system. Under this approximation, a modified Lorenz model is derived [13]. The latter model reads as:

$$\dot{X} = \sigma(rY(1 + Z) - X) \quad (1)$$

$$\dot{Y} = X(1 - Z) - Y, \quad (2)$$

$$\dot{Z} = XY - 2Z, \quad (3)$$

where the dot ($\cdot$) denotes the time derivative $d()/dt$. From the physical point of view, X is related to the velocity field, whereas $\{Y, Z\}$ are related to the concentration field. In addition, σ is the inverse Schmidt number, namely the ratio of momentum diffusivity to mass diffusivity, while r is the activity Rayleigh number. The factor YZ is the leading-order active coupling retained by the Galerkin truncation. Since Y and Z encode concentration modes, the contribution σrYZ feeds back on the velocity mode X and can be interpreted as the reduced active-stress feedback of concentration inhomogeneities on the flow. Dynamically, this term does not merely shift thresholds. It modulates the effective forcing σrY by the instantaneous state of Z , thereby altering the location of bifurcations and the geometry of the periodic and chaotic attractors.

At the same time, the system remains strictly dissipative: evaluating the divergence of the flow,

$$\nabla \cdot \vec{V} = \frac{\partial \dot{X}}{\partial X} + \frac{\partial \dot{Y}}{\partial Y} + \frac{\partial \dot{Z}}{\partial Z} = -(\sigma + 3), \quad (4)$$

shows that phase-space volumes contract at a constant rate, exactly as in Lorenz-type dissipative flows. The active nonlinear terms reorganize the internal geometry of the trajectories, but they do not modify this global contraction rate. Unlike passive Lorenz-type reductions derived from conservative hydrodynamic limits, the present active three-mode model does not admit a physically controlled dissipationless limit within the same reduced variables. Formally suppressing the linear damping terms makes the divergence vanish, but this operation does not reconstruct a conservative active-fluid dynamics, nor does it recover a known invariant structure of the parent hydrodynamic equations. The reason is that the Galerkin truncation has already eliminated spatial degrees of freedom, while the active feedback remains intrinsically nonequilibrium. For this reason, we interpret the present system as a dissipative active flow for all physically meaningful parameter regimes considered here.

A direct one-to-one parameter mapping to the classical Lorenz equations is therefore not available globally. The cleanest comparison is local, near the origin, where the active contribution rYZ vanishes to leading order and the difference reduces to the linear coupling structure. Away from that small-amplitude regime, however, the extra quadratic feedback modifies the nonlinear closure itself, so no global parameter choice reproduces the classical Lorenz dynamics exactly. The geometric parameter of the classical Lorenz equations is replaced here by the fixed value $b = 2$, imposed by the hydrodynamic projection [20]. Another Lorenz-type model with nonlinear terms in all three equations was derived in the context of ferrofluid convection under an applied magnetic field [28]; in that case the standard Lorenz model is recovered when the magnetic field is switched off, whereas in Eqs. (1)–(3) the active feedback is intrinsically present for $r \neq 0$. The system nevertheless preserves the main Lorenz symmetry $(X, Y, Z) \mapsto (-X, -Y, Z)$.

The transition from stationary to chaotic states, along with the primary Hopf bifurcation, was analyzed recently in Ref. [14]. Additionally, the model has been studied for negative values of σ [13]. Because this formulation is a minimalist Galerkin truncation over leading modes, it captures global temporal mode selection but filters out spatial textures, orientational degrees of freedom, and noise-driven switching. The results presented below should therefore be interpreted as a low-dimensional dynamical characterization of the reduced active system, not as a full substitute for the parent hydrodynamic fields. However, to the best of our knowledge, a comprehensive characterization of the model is still lacking. In the following section, we examine various dynamical indicators as functions of the parameters (σ, r) and of the initial conditions. Due to the multiple-stability, we will show that the basins of attraction reveal interesting complex patterns.

2.2. Numerical approach

The system of Eqs. (1)–(3) is numerically integrated using a standard fourth-order Runge-Kutta method with a fixed time step $\Delta t = 0.001$. Transient behavior are always discarded by removing the first 5×10^6 steps in real space, and 3×10^4 normalization intervals in the tangent space while computing the Lyapunov exponents. For the latter, orthonormalization is performed every 100 steps using the Gram-Schmidt algorithm. Isospike and complexity analyses are performed after an additional transient of 3×10^6 steps has elapsed. A fixed random seed ensures reproducibility of all the computations. This numerical protocol allows reliable simulation of long-term dynamics and supports precise detection of dynamical transitions in the parameter space.

2.3. Dynamical indicators and quantitative characterization

To explore the complex dynamical behavior of the system defined by Eqs. (1)–(3), we employ a combination of complementary indicators: the largest Lyapunov exponent (LLE), isospike analysis, the López-Ruiz-Mancini-Calbet (LMC) statistical complexity, and the time-averaged value of dynamic variables as a scalar classifier of coexisting attractors.

The largest Lyapunov exponent quantifies the average rate of separation of infinitesimally close trajectories in phase space. Given a small initial perturbation $\delta \mathbf{X}(0)$ to a reference trajectory $\mathbf{X}(t)$, the LLE is computed as

$$\lambda = \lim_{t \rightarrow \infty} \left(\frac{1}{t} \ln \left(\frac{\|\delta \mathbf{X}(t)\|}{\|\delta \mathbf{X}(0)\|} \right) \right), \quad (5)$$

where $\delta \mathbf{X}(t)$ evolves according to the linearized tangent flow. A positive λ indicates chaotic state, while $\lambda = 0$ and $\lambda < 0$ correspond to periodic (or quasi-periodic) and stationary states, respectively. We compute the LLE using the standard Benettin algorithm with periodic Gram-Schmidt orthonormalization every 100 integration steps [21,22,29].

To resolve fine structures in the parameter space and distinguish periodic regimes, we construct isospike diagrams by counting the number of local maxima (spikes) in the time series of each component of Eqs. (1)–(3). Each constant-spike region corresponds to a periodic orbit with well-defined spike count, while sudden changes in this count signal bifurcations, transitions to chaos, or switching between coexisting attractors. This methodology allows the construction of detailed periodicity maps and facilitates the identification of singular structures such as period-adding cascades and quintuplet points [24,30].

To quantify the balance between randomness and order in the time series, we evaluate the LMC complexity [23], defined as the product of the normalized Shannon entropy (H) and the disequilibrium (D), computed from symbolic dynamics:

$$C_{\text{LMC}} = H \cdot D \quad (6)$$

where H and D are mathematically defined as

$$H = -k \sum_{i=1}^N p_i \log p_i, \quad (7)$$

$$D = \sum_{i=1}^N \left(p_i - \frac{1}{N} \right)^2, \quad (8)$$

such that p_i are the empirical probabilities of symbolic patterns of length n_s , obtained by coarse-graining the time series through binarized and thresholding schemes [23]. In these definitions, k denotes a positive constant and N represents the maximum number of accessible states. The algorithm used to compute C_{LMC} must be iterated until the indicator becomes invariant with respect to the length n_s . In our case, we found that $n_s = 12$ ensures the convergence of the algorithm. This method has been extensively applied across a wide variety of dynamical systems [31,32]. It is worth noting that the complexity indicator vanishes in regular regimes (periodic or stationary), while it takes strictly positive values in chaotic states.

Finally, to characterize the fragmented basins of attraction and identify coexisting attractors under identical control parameters, we also compute the time-averaged value of dynamic observables such as $X(t)$ or $Y(t)$, defined generically as

$$\langle \Omega \rangle = \frac{1}{T} \int_{t_0}^{t_0+T} \Omega(t) dt. \quad (9)$$

Let us note that the time average is only calculated after removing a long transient, here denoted by t_0 . This scalar quantity serves as a compact descriptor of the attractor reached from each initial condition. Following the approach of Ref. [33], we map $\langle X \rangle$ over a two-dimensional grid of initial conditions to visualize the fine structure of multistable basins. This approach enables the discrimination of coexisting attractors based on their geometric or statistical properties, even when they exhibit similar spectral features or share the same qualitative behavior (e.g., both being chaotic). As demonstrated in Ref. [33], this method provides a reliable and computationally efficient tool for resolving complex basin fragmentation patterns in nonlinear systems.

Additionally, to quantify the degree of basin fragmentation, we compute the Basin Entropy S_b introduced by Daza et al. [34]. This metric partitions the initial-condition grid into boxes of linear size ε (in grid units) and evaluates the average Shannon entropy of the attractor distribution within each box:

$$S_b = \frac{1}{N_{\text{box}}} \sum_{i=1}^{N_{\text{box}}} S_i, \quad S_i = - \sum_{j=1}^{N_A} \eta_j \log \eta_j, \quad (10)$$

where η_j is the fraction of points in box i converging to attractor j , and N_A is the number of coexisting attractors. A value $S_b = 0$ indicates completely separated basins, whereas $S_b = \ln N_A$ corresponds to maximally intertwined basins. Computing S_b over a range of box sizes ε enables a multi-scale diagnosis of basin fragmentation; persistent positive values across scales indicate that the basins remain intermixed under refinement.

Together, these tools provide a robust and complementary framework to map the full dynamical landscape of the system, resolve transitions across bifurcations, and uncover the rich multistable and singular structures present in the parameter space. Finally, let us comment that unless otherwise specified, the following two-dimensional maps for the different dynamical indicators are constructed on an equidistant grid of 2000×2000 points.

3. Global structure in parameter space and routes to chaos

In this section, we investigate the global organization of the dynamical regimes in the (σ, r) parameter space using a combination of complementary indicators: the largest Lyapunov exponent, isospoke analysis, and the LMC statistical complexity. By systematically scanning the control space and analyzing the resulting trajectories, we uncover a rich landscape of dynamical behavior, including periodic windows, chaotic bands, multistability, and singular bifurcation structures. The combined use of these tools provides both a quantitative and geometric perspective on the transitions between dynamical states and allows for a robust classification of the underlying attractors.

Fig. 1 shows a detailed exploration of the parameter space (σ, r) using several dynamical indicators. These maps were obtained using the initial conditions $X_0 = Y_0 = Z_0 = 0.9$, and they reveal a rich variety of dynamical regimes ranging from fixed points to periodic and chaotic attractors. Panel Fig. 1(a) displays the maximum Lyapunov exponent, which clearly identifies the chaotic regions (colored from yellow to red) where the exponent is positive, in contrast to regular regimes (black), where the exponent is zero or negative. This panel provides a global view of the dynamical instabilities as both σ and r increase, revealing wide bands of chaos interlaced with ordered dynamics. These chaotic bands are typically aligned with known bifurcation structures, particularly regions of period-doubling and crises. In Panel Fig. 1(b), the distribution of dynamical periods is shown, obtained from spike counting in the time series. Periodic regimes are color-coded by the number of spikes per period (from cyan to blue), whereas chaotic states appear in black, and fixed points in white. This panel captures fine-grained bifurcation details, including period-doubling cascades and windows of periodicity embedded in chaotic regions. The abrupt transitions in color highlight the system's sensitivity to parameter changes, often leading to the coexistence of multiple periodic or chaotic states in narrow parameter intervals. Panel Fig. 1(c) presents the complexity indicator, which quantifies the degree of structural richness in the time series. As expected, complexity is low (white) in fixed points and periodic orbits, and high (blue) in chaotic regimes, where unpredictability and aperiodic oscillations dominate. Interestingly, some regions of moderate complexity are found in transitions between order and chaos, potentially corresponding to quasi-periodic or intermittency regimes.

The isospoke diagrams shown in Panels Fig. 1(d)–(f) offer complementary insights by tracking the number of peaks in the variables $X(t)$, $Y(t)$, and $Z(t)$, respectively. These maps provide a visual fingerprint of the oscillatory dynamics and help distinguish between different periodic regimes. Black regions represent chaotic behavior, where no regular spike count can be defined. Despite similarities among the three variables, subtle differences in spike structure reflect the internal coupling and nonlinear interactions between the components of the system. Such differences are especially relevant near bifurcation lines and chaotic boundaries. Taken together, these indicators reveal the intricate topology of the parameter space in the driven active matter model. The coexistence of multiple attractors, the presence of sharp transitions, and the fragmentation of periodic windows within chaotic bands point to a system with a high degree of multistability and dynamical richness. The combined use of Lyapunov exponents, spike counts, and complexity metrics enables a more robust classification of the observed behavior, as previously demonstrated in related studies of nonlinear systems.

Fig. 1 should be read as a global concordance map of the parameter plane. The Lyapunov, period, complexity, and isospoke panels were computed on the same (σ, r) grid and are displayed with identical axis limits for direct visual comparison. Because the period and isospoke panels resolve periodic substructure, whereas the Lyapunov and LMC indicators separate chaos from order, the most

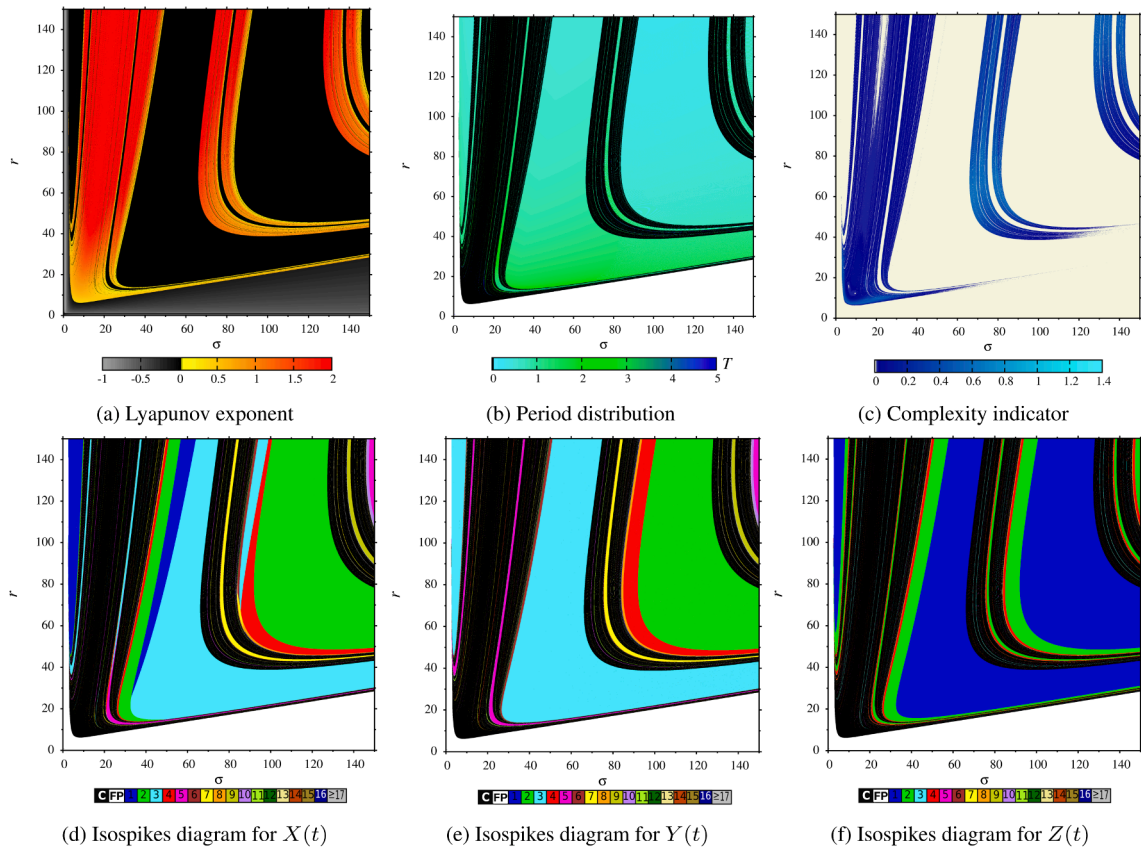


Fig. 1. Dynamical indicators computed over the parameter space (σ, r) using initial conditions $X_0 = Y_0 = Z_0 = 0.9$. (a) Maximum Lyapunov exponent: Black indicates non-chaotic regimes (periodic or fixed points), while yellow to red represents chaos. (b) Period distribution: Periodic regimes are shown in cyan-blue scale, chaos in black, and fixed points in white. (c) LMC complexity indicator: High values (blue) indicate chaotic complexity; white corresponds to simple dynamics. (d–f) Isospikes diagrams showing the number of local maxima in $X(t)$, $Y(t)$, and $Z(t)$ respectively. Color maps reveal changes in spike structure across parameter space; black indicates chaotic oscillations with irregular spike sequences. All panels are displayed with the same axis limits to facilitate direct visual comparison.

meaningful quantitative comparison is performed along a common one-dimensional cut. For this reason, the representative cut $r = 100$ is analyzed explicitly in Fig. 3, and the four states discussed in Fig. 4 are selected from that same interval.

Fig. 2 provides a zoomed-in view of a specific region in the (σ, r) parameter space previously shown in Fig. 1. It uses the same initial conditions and dynamical indicators to enable a more detailed analysis of the transitions between different dynamical regimes. Panel Fig. 2(a) shows the maximum Lyapunov exponent. In this magnified view, bifurcations and transitions to chaos (yellow to red areas) can be identified with greater clarity. Black regions correspond to non-chaotic behavior, where the system converges to periodic or fixed-point solutions. This localized map highlights the chaotic pockets within the selected zone of parameter space. Panel Fig. 2(b) presents the period distribution. As in the Lyapunov map, chaotic regions appear in black, indicating the loss of periodicity. Cyan to blue tones represent periodic states with different numbers of oscillations. The transition between periodic and chaotic regimes reinforces the bifurcation structure identified earlier. Panel Fig. 2(c) displays the complexity indicator. In this localized frame, regions of high complexity (blue) correlate with those of positive Lyapunov exponent, while low-complexity regions (lighter tones) indicate regular or stationary dynamics. This panel complements the others by providing insight into the internal structure of the trajectories and their sensitivity to parameter variation.

Panels Fig. 2(d)–(f) show isospikes diagrams for the X , Y , and Z components. These diagrams display the number of peaks in each variable's time series as a function of σ and r . Colored regions correspond to well-defined periodic oscillations, while black areas mark chaotic behavior where peak sequences are irregular. The diagrams reveal how oscillatory structures vary with parameters and highlight subtle differences among variables, despite overall dynamic coherence. The three main indicators, Lyapunov exponent, period distribution, and complexity, complement each other, offering a consistent picture of the system's transitions in this refined region. Chaotic zones identified by positive Lyapunov exponents coincide with regions of high complexity, suggesting irregular, sensitive trajectories. Meanwhile, the isospikes diagrams reinforce this picture by capturing how the number of oscillations per cycle changes with parameters. In particular, the spike counts vary in synchrony across X , Y , and Z , indicating that transitions affect the entire system rather than individual variables in isolation. This detailed analysis provides a deeper understanding of how the system

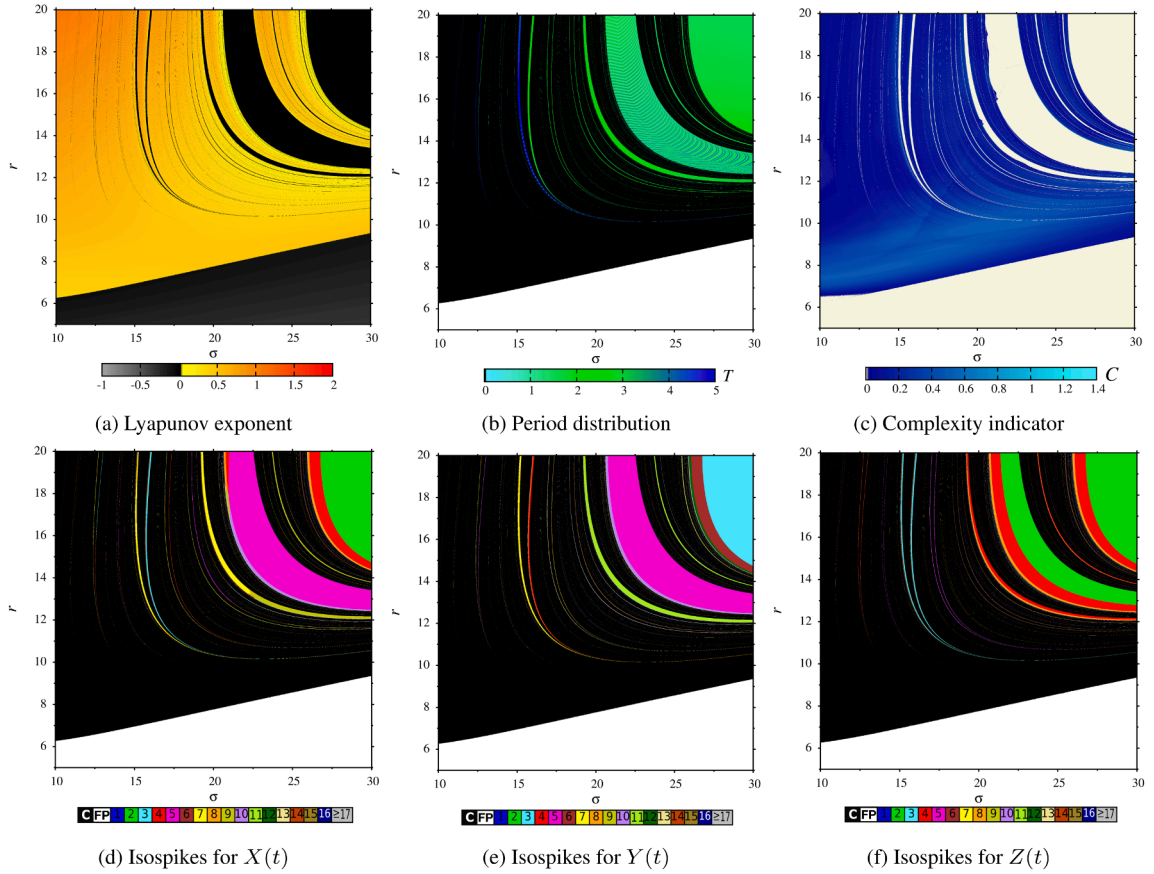


Fig. 2. Zoomed-in view of a selected region from Fig. 1, using the same initial conditions and dynamical indicators. Each panel represents a different indicator: (a) Maximum Lyapunov exponent, (b) Period distribution, (c) Complexity indicator, and (d–f) isospike diagrams for the X , Y , and Z components, respectively. This figure allows for detailed observation of the transitions between dynamical regimes in a specific region of the parameter space (σ, r) .

transitions between periodic and chaotic regimes within a focused region of parameter space. It reveals the internal structure of the trajectories and how their characteristics evolve with changes in σ and r .

3.1. Transitions from order to chaos

Fig. 3 presents a comparative analysis of the dynamical states of the active matter system using two key indicators, the LMC complexity and the maximum Lyapunov exponent λ_1 , alongside a bifurcation diagram of the Z component as a function of the control parameter σ . Fig. 3(a) displays the bifurcation diagram of Z for $r = 100$ as σ varies. The diagram reveals a sequence of transitions between periodic and chaotic regimes, including classical period-doubling bifurcations that mark the onset of chaos. These bifurcations are particularly evident around $\sigma \approx 40$, where the system shifts from stable periodic orbits to increasingly irregular trajectories, a hallmark of nonlinear systems exhibiting deterministic chaos. Fig. 3(b) and (c) provide a detailed zoom of the red box from Fig. 3(a), comparing the values of LMC complexity (red curve) and the maximum Lyapunov exponent (blue curve) over the same parameter interval. An additional bifurcation diagram is included at the top of panel (c) to visually align these indicators with the underlying trajectory transitions. The quantitative comparison in Fig. 3(c) is performed on the common cut $r = 100$ over the plotted interval $39.85 \leq \sigma \leq 50.0$. Using the thresholds $\lambda_1 > 0.005$ and $C_{\text{LMC}} > 0.01$, both indicators yield the same chaotic/non-chaotic classification for 1670 of the 1692 sampled points (98.7%). The remaining 22 points (1.3%) are confined to a narrow transitional region, approximately $40.844 \leq \sigma \leq 41.780$, where λ_1 remains very close to zero and the complexity signal is more sensitive to incipient irregular peak statistics. We therefore interpret these discrepancies as threshold-sensitive transitional points rather than contradictory classifications. Sharp peaks in both complexity and λ_1 mark the locations of bifurcations and the emergence of chaos. These peaks align with significant changes in the structure of the attractor, as observed in the bifurcation diagrams. Notably, the consistency between the indicators reinforces the interpretation of the observed transitions, and their combined use provides a more reliable characterization of dynamical changes.

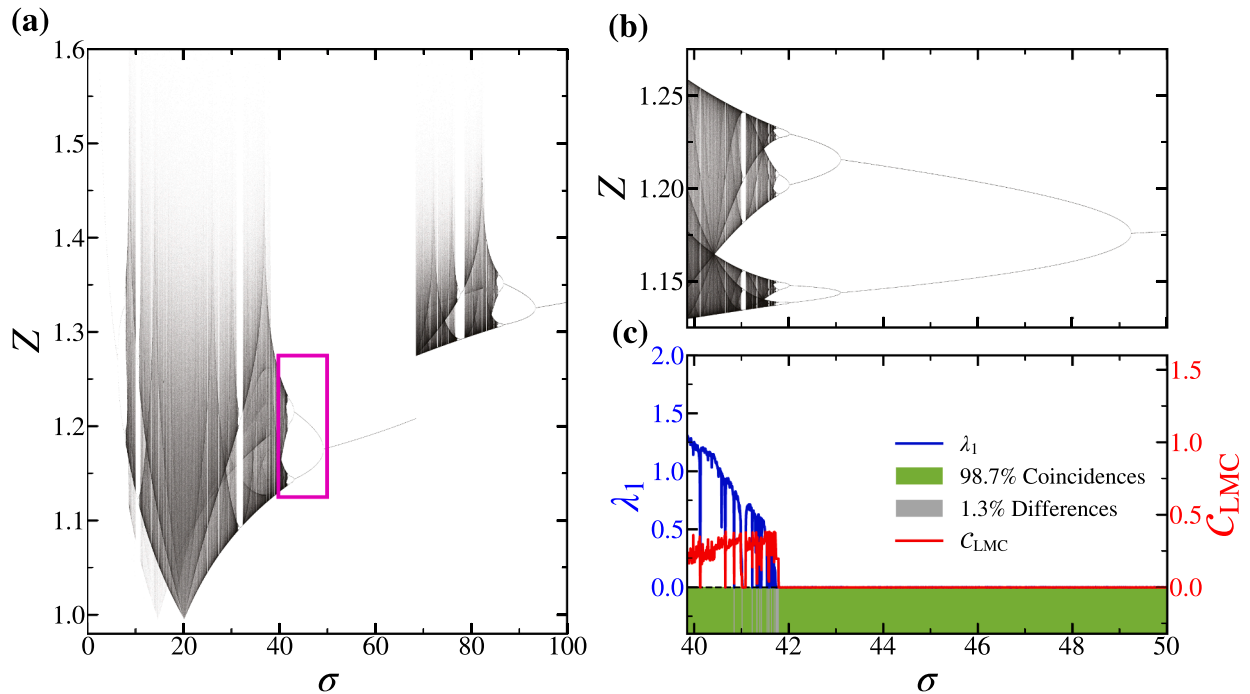


Fig. 3. (a) Bifurcation diagram of the Z component as a function of σ with $r = 100$, showing transitions from periodic to chaotic dynamics through period-doubling bifurcations. (b) Zoomed-in bifurcation diagram over the focal interval $39.85 \leq \sigma \leq 50.0$, with markers A–D indicating the four representative states analyzed in Fig. 4. (c) Detailed comparison of the LMC complexity (red) and the maximum Lyapunov exponent λ_1 (blue) over that same interval. Over the plotted cut, both indicators agree for 1670 of the 1692 sampled points (98.7%), whereas the remaining 22 points (1.3%) are confined to a narrow transitional strip.

Fig. 4 presents a comparative view of four dynamical states sampled from the bifurcation diagram shown in Fig. 3(b). For each selected value of the control parameter σ , we display the corresponding time series, Fourier spectrum, and phase-space projection (attractor). In all cases, the parameter $r = 100$ is held fixed. Panel (a) shows the system dynamics for $\sigma = 40$, where chaotic behavior is clearly observed. The time series is irregular, with no repeating pattern in amplitude or frequency. The Fourier transform reveals a broad, continuous spectrum typical of chaotic systems, where many frequencies overlap. The phase portrait further confirms the presence of a chaotic attractor, characterized by an aperiodic, non-repeating trajectory in state space. Panel (b), corresponding to $\sigma = 41.9$, exhibits a transition from chaos to a periodic regime. The time series becomes more regular and oscillatory, indicating the emergence of a limit cycle. The Fourier transform shows well-defined peaks, characteristic of dominant frequencies in periodic motion. The attractor has reorganized into a closed loop in phase space, reflecting stable periodic behavior. In Panel (c), at $\sigma = 45$, the periodic behavior persists. The time series remains oscillatory, with minor amplitude modulations. The Fourier spectrum retains its sharp peaks, and the attractor becomes slightly more intricate, though it remains periodic in nature. Panel (d), corresponding to $\sigma = 50$, continues to display periodic dynamics. The time series and Fourier spectrum show strong regularity, although the amplitude and spacing of frequency peaks have changed, suggesting variations in the internal frequency content. The attractor appears more geometrically complex but still follows a closed, repetitive orbit. These four cases are fully consistent with the quantitative indicators along the same cut: point A is chaotic, with $\lambda_1 \approx 1.252$ and $C_{\text{LMC}} \approx 0.214$, whereas points B, C, and D belong to non-chaotic windows with λ_1 very close to zero and vanishing complexity. The transition from A to D therefore provides a direct visual realization of the same order-chaos classification quantified in Fig. 3(c). It is noteworthy that, while the system remains in a periodic regime for increasing values of σ , the structure of the attractors becomes progressively more elaborate. This increase in geometric complexity suggests a deepening of the internal organization of the dynamics, despite the persistence of periodicity. The transition from chaos at $\sigma = 40$ to periodic states at $\sigma = 41.9$, $\sigma = 45$, and $\sigma = 50$ illustrates a clear route toward order. These transitions are consistent with the bifurcation structure identified in the zoomed bifurcation diagram of Fig. 3(b), reinforcing the interpretation of an inverse route to chaos through periodic stabilization.

4. Quintuplet points and local structures

The global indicators presented in Section 3 revealed a rich (σ, r) parameter landscape, including transitions between order and chaos as well as clear evidence of attractor coexistence. To further elucidate the local mechanisms underlying these transitions, we now turn to a fine-resolution analysis based on *isospike* structures and temporal dynamics. This approach allows us to identify

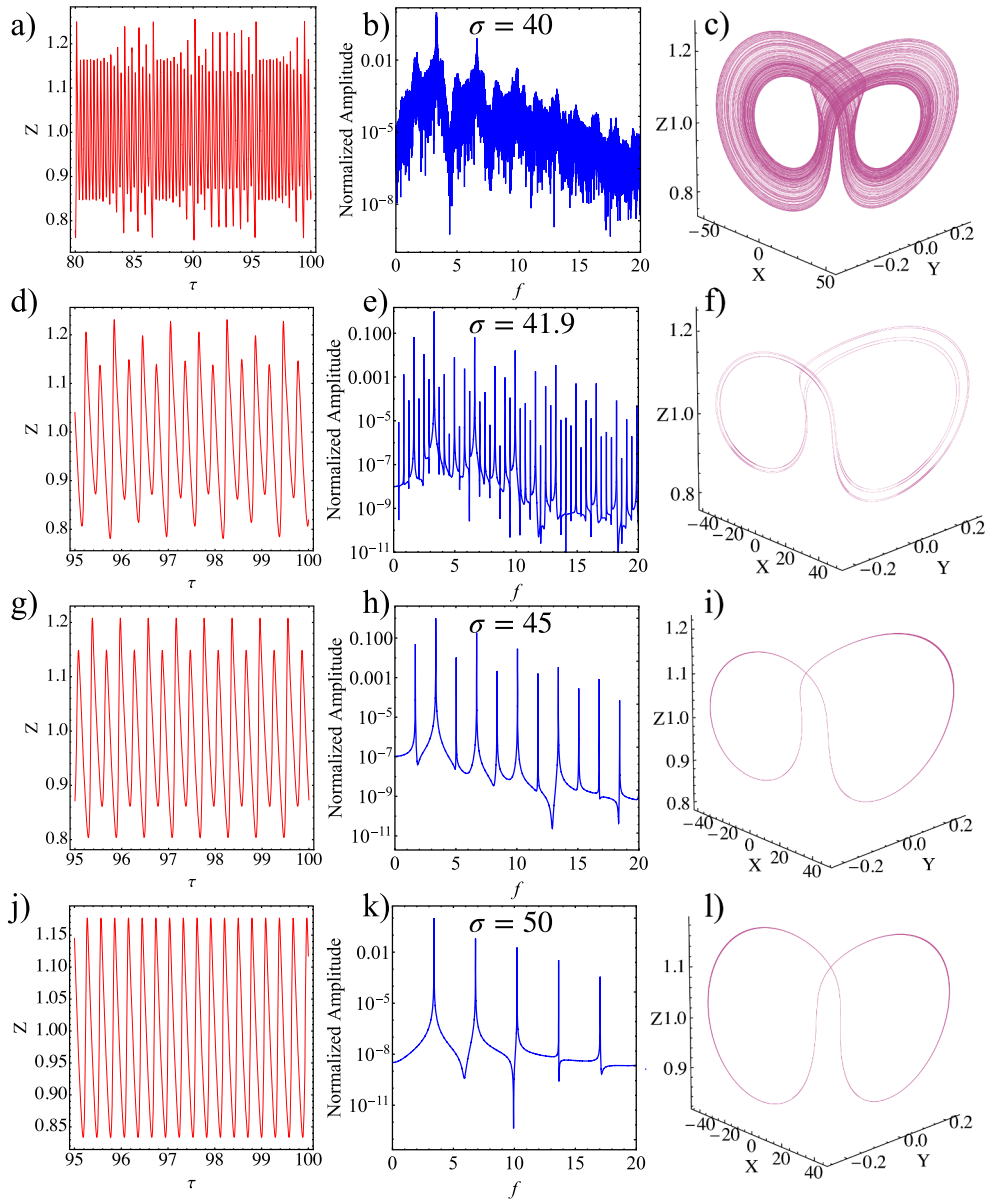


Fig. 4. Simulation results for four different dynamical states of the system. Each row displays the time series (left), Fourier spectrum (center), and phase-space attractor (right) for a fixed value of $r = 100$ and different values of σ : (a) chaotic state ($\sigma = 40$), (b) periodic state ($\sigma = 41.9$), (c) periodic state ($\sigma = 45$), and (d) periodic state ($\sigma = 50$). The four rows correspond to the points A–D marked in Fig. 3: A ($\sigma = 40.0$), B ($\sigma = 41.9$), C ($\sigma = 45.0$), and D ($\sigma = 50.0$).

the organizing features that govern the emergence and interaction of periodic regimes, providing a detailed picture of how local bifurcations shape the complex structure of the control space.

Fig. 5 reveals a particularly intriguing feature, namely a quintuplet point located at the intersection of five distinct periodic regions. This type of singular point, recently identified in chemically and mechanically driven nonlinear systems [24,35,36], represents a highly organized local structure where bifurcations align in a combinatorial pattern. In our system, this quintuplet point emerges in a reduced Lorenz-like model for active matter, marking, to our knowledge, the first observation of such a structure in this class of models. Panel (b) shows that around the quintuplet point, spike-adding bifurcations organize into arcs and cascades, resulting in a rich variety of coexisting periodicities. Each domain corresponds to a specific number of local maxima in $X(t)$ per oscillatory cycle, and their alignment suggests an underlying regularity governed by local rules of spike addition. The formation of such structures reflects a form of discrete organization in the bifurcation topology, reminiscent of those found in other nonlinear dissipative systems. Within the present minimal active-matter reduction, the quintuplet point should therefore be interpreted as a local organizer of mode

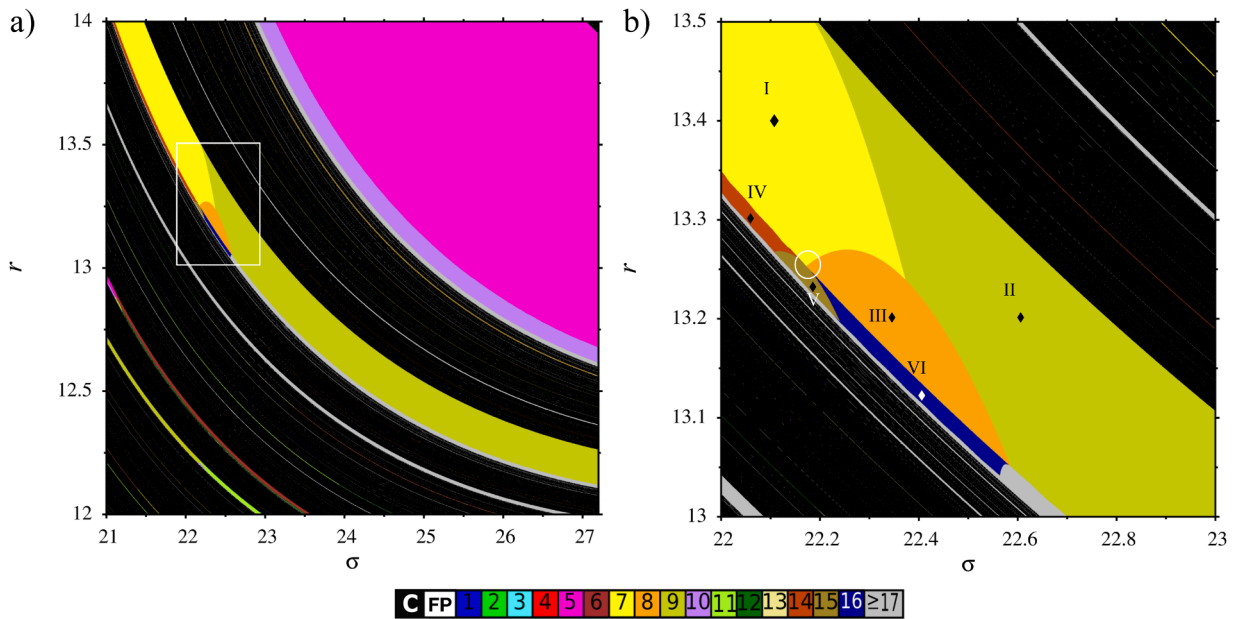


Fig. 5. (a) Zoomed-in isospike diagram of the $X(t)$ component in the (σ, r) parameter plane, showing a region with complex transitions between different periodic regimes. (b) Enlargement of the highlighted area in (a), where a quintuplet point is identified: a singular structure where five distinct periodic domains coalesce at a single location in parameter space, marked by a white circle. Regions labeled with Roman numerals (I–VI) indicate specific transitions that are analyzed in detail in the following figure.

Table 1

Control parameters, spike count (isospike), and period for the six representative points around the quintuplet structure shown in Fig. 5. Periods are the fundamental oscillation periods obtained from the time series illustrated in Fig. 6.

Region	r	σ	Isospike	Period T
I	13.40	22.10	7	7.7446
II	13.20	22.60	9	7.7446
III	13.20	22.40	8	7.8513
IV	13.30	22.08	14	15.5952
V	13.25	22.15	15	15.6379
VI	13.12	22.40	16	15.7896

selection and spike reorganization in the coupled velocity-concentration dynamics, rather than as a direct one-to-one experimental observable.

Fig. 6 shows representative time series associated with the six regions labeled (I–VI) around the quintuplet point identified in Fig. 5. In cases I, II, and III, the trajectories exhibit a similar temporal period ($T \approx 7.7$), highlighted in pink, but with different spike counts: 7, 9, and 8, respectively. The control parameters, isospike counts, and periods for the six cases are summarized in Table 1.

This variation suggests a transitional phase where small parameter changes lead to internal reorganizations of the oscillatory structure without altering the overall cycle duration. In contrast, regions IV, V, and VI exhibit a sudden change: the period doubles ($T \approx 15.6$), and the number of spikes increases accordingly (14, 15, and 16), maintaining a clear correspondence with the previous cases. This behavior indicates the occurrence of a period-doubling bifurcation accompanied by spike-doubling, consistent with the band structure observed in the isospike diagram. The transition between bands marks a qualitative change in the system's dynamics, where the oscillatory orbit requires twice the time to close and becomes more intricate. Within the diagnostics used here, the quintuplet structure is identified most sharply by the isospike map, but it is also supported by the time-series and period analysis in Fig. 6 and Table 1. Regions I–III retain approximately the same fundamental period while reorganizing the number of spikes, whereas regions IV–VI display a near doubling of both period and spike count. We therefore interpret the quintuplet point as a local organizer of spike-adding and spike-doubling transitions in the two-parameter plane. Because the term rYZ is the only quadratic contribution absent from the classical Lorenz equations, it is the natural mechanism by which activity deforms the local bifurcation geometry. However, the present study does not isolate that term independently, so we attribute the observed quintuplet point to the full active reduced system rather than claiming that YZ alone is sufficient to generate it. In this sense, the quintuplet point should be viewed as

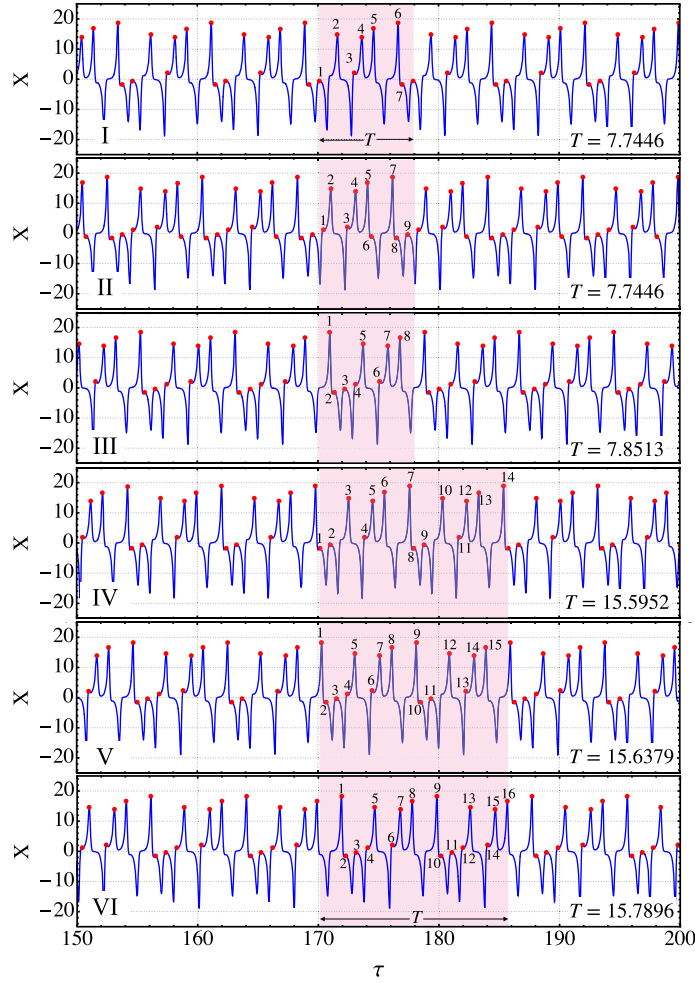


Fig. 6. Representative time series of $X(t)$ corresponding to the six labeled regions (I-VI) around the quintuplet point shown in Fig. 5. Panels I-III display oscillatory states with similar periods ($T \approx 7.7$) but distinct spike counts (7, 9, and 8), indicating internal reorganization of the trajectory structure. In contrast, panels IV-VI show a doubling of the temporal period ($T \approx 15.6$) along with an increased number of spikes (14, 15, and 16), consistent with a spike-doubling bifurcation across adjacent isospike bands. This transition reveals a qualitative change in the oscillatory dynamics, highlighting the role of the quintuplet point as a local organizer of bifurcation sequences.

a reduced-model signature that small changes in activity control can reorganize oscillatory mode selection in the coupled velocity-concentration dynamics.

4.1. Recurrence rules at the quintuplet point

The spike-count data in Table 1 are not arbitrary; they obey a set of *recurrence rules* that have been identified in other dissipative systems [24,35,37,38]. Specifically, at the quintuplet point where the boundary of two adjacent periodic phases with spike counts n_1 and n_2 terminates, three new daughter phases emerge: one via spike addition (a Farey-like sum $n_{\text{add}} = n_1 + n_2$) and two via spike doubling ($n_{d,1} = 2n_1$ and $n_{d,2} = 2n_2$). Together with the parent phases, these three daughter phases produce the five domains that coalesce at the quintuplet point, with the doubling accompanied by a corresponding period doubling $T_{\text{doubled}} \approx 2T_{\text{parent}}$. In all the cases, one can see that $n_1 < n_2$. In our previous simulations, the parent phases have $n_1 = 7$ (Region I) and $n_2 = 8$ (Region III). Applying these rules yields $n_{\text{add}} = 7 + 8 = 15$ (Region V, observed: 15 spikes), $2n_1 = 14$ (Region IV, observed: 14 spikes), and $2n_2 = 16$ (Region VI, observed: 16 spikes). This feature is in exact agreement with Table 1, in which the period doubling is confirmed by $T_{\text{IV-VI}} \approx 15.6 \approx 2 \times 7.8 = 2T_{\text{I-III}}$.

Appendix A provides a detailed derivation of the recurrence rules, as well as new high-resolution parameter sweeps in two additional regions of the (σ, r) plane, confirming that the same recurrence rules appear in the isospike cascades.

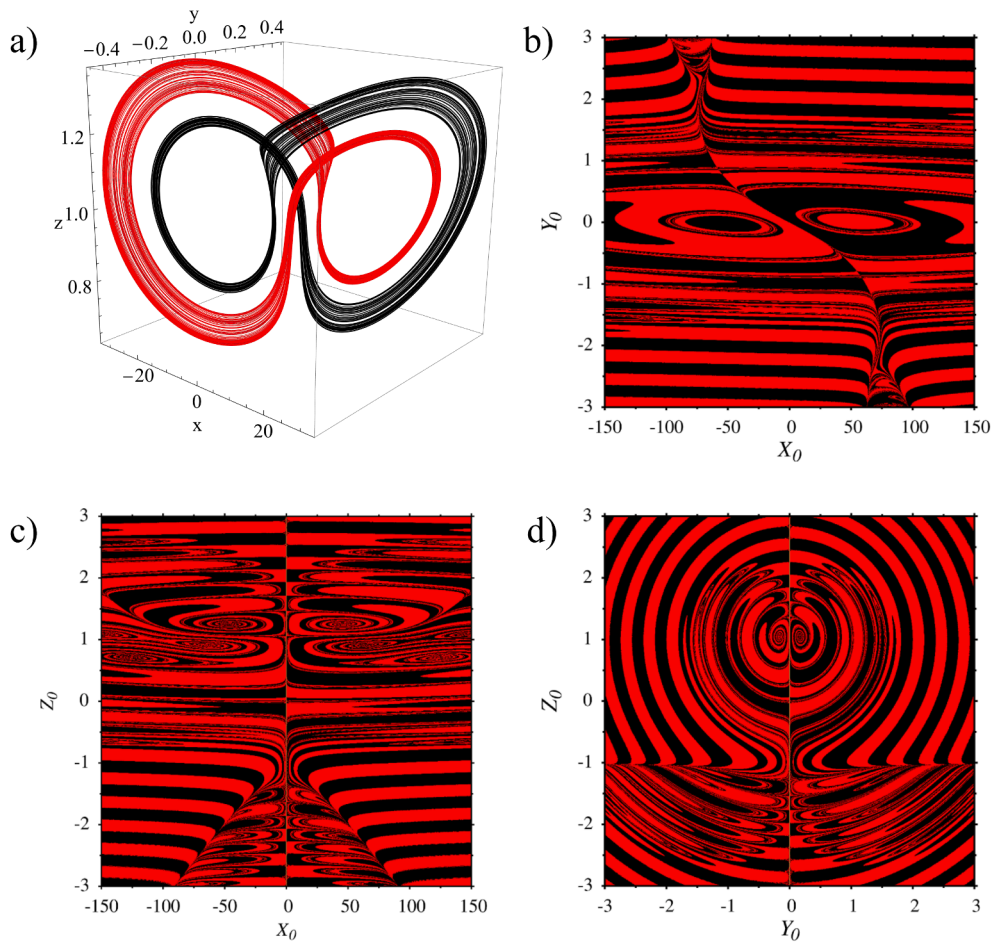


Fig. 7. Coexistence of two distinct chaotic attractors for the same control parameters $\sigma = 30$ and $r = 43$, obtained with different initial conditions. (a) 3D trajectories of the two coexisting attractors, shown in red and black, respectively. (b-d) Maps showing which attractor is selected depending on the specified initial conditions: (b) X_0 vs. Y_0 ($Z_0 = 0.5$), (c) X_0 vs. Z_0 ($Y_0 = 0$), (d) Y_0 vs. Z_0 ($X_0 = 0$). All the regions shown in this figure have positive Lyapunov exponents, confirming chaotic dynamics throughout.

5. Multistability and basin fragmentation

One of the most striking features of our model is this global multistability under identical control parameters, which implies that the long-term behavior is not uniquely determined by the parameters but depends sensitively on the initial conditions [25]. This interplay between local bifurcation geometry and global dynamics naturally leads to the emergence of multiple coexisting attractors separated by fragmented basins. Here the relevant sensitivity is primarily final-state sensitivity in a multistable setting: nearby initial conditions may converge to different asymptotic attractors separated by fragmented basins. This notion is complementary to, rather than a replacement for, the standard sensitivity to initial conditions quantified by positive Lyapunov exponents within a single chaotic attractor. The present model displays both mechanisms, but they enter the analysis in different ways. Such a scenario invites a natural comparison with the classical Lorenz system, where metastable chaos and coexistence of chaotic and non-chaotic attractors have also been reported [39,40]. In contrast, the attractors reported here are resolved as distinct asymptotic states selected by the initial conditions at fixed parameter values, as documented by the basin scans in Figs. 7 and 8. In particular, we report here the coexistence of distinct chaotic attractors, as well as the coexistence of chaos and fixed points. These results are not only significant in the context of nonlinear dynamics but also highly relevant for physical systems such as active matter, where external fluctuations can easily perturb the initial state, triggering transitions between qualitatively different behaviors.

Figs. 7 and 8 illustrate the coexistence of distinct attractors under the same set of control parameters, emphasizing the strong dependence of the system on initial conditions. These results provide compelling evidence of multistability and complex basin structures in the phase space of the model. Fig. 7 shows the coexistence of two different chaotic attractors for the parameters $\sigma = 30$ and $r = 43$. Panel 7(a) displays the 3D trajectories corresponding to two different initial conditions, where each trajectory converges to a different chaotic attractor, represented in red and black, respectively. Despite both states being chaotic (confirmed by a positive Lyapunov exponent), their geometric structure and average values differ, demonstrating true dynamic coexistence. Panels 7(b), (c),

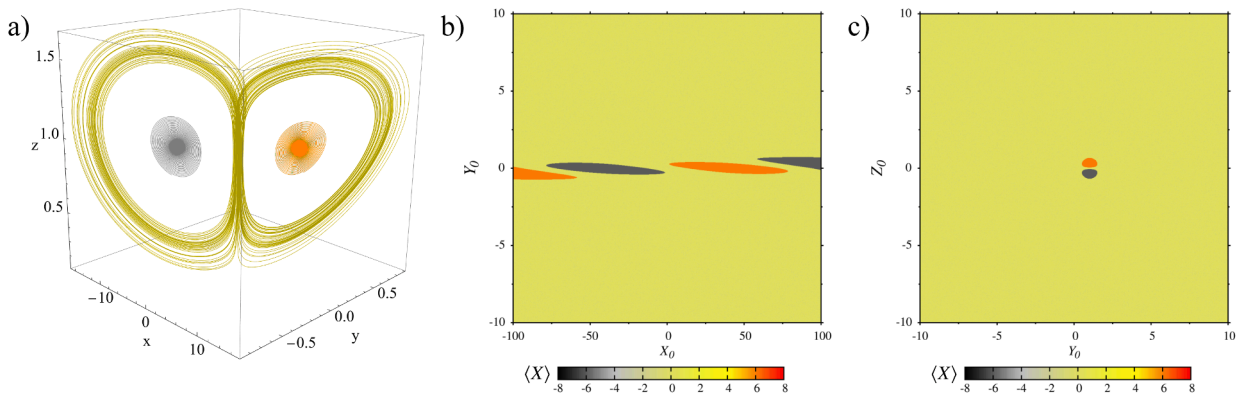


Fig. 8. Multistability between two fixed points and one chaotic attractor for $\sigma = 30$ and $r = 10$. (a) Time-averaged X value $\langle X \rangle$ distinguishes the three attractors. (b) Initial condition scan over X_0 and Y_0 ($Z_0 = 0.9$). (c) Scan over Y_0 and Z_0 ($X_0 = 0.5$). Small variations in initial conditions cause transitions between attractors.

and (d) explore the transition between these attractors by systematically varying pairs of initial conditions. In panel (b), X_0 and Y_0 are varied with $Z_0 = 0.5$ fixed; in panel (c), X_0 and Z_0 are varied with $Y_0 = 0$; and in panel (d), Y_0 and Z_0 are varied with $X_0 = 0$. In all cases, the phase space is partitioned into basins of attraction for each chaotic state. The color-coding reflects which attractor is reached, while the positive Lyapunov exponent remains constant across the entire domain, indicating that all accessible attractors are chaotic. This implies that the system exhibits not only multistability but also *chaotic multistability*, where multiple chaotic solutions coexist and compete in the phase space depending on the initial conditions.

In contrast, Fig. 8 presents a scenario where the system exhibits coexistence between two fixed points and a chaotic attractor for $\sigma = 30$ and $r = 10$. Panel (a) shows the time-averaged value $\langle X \rangle$ for each attractor, allowing to distinguish between the two fixed-point states characterized by constant $\langle X \rangle$ values (one positive and one negative) and the chaotic state (characterized by fluctuating $\langle X \rangle$ around zero).

Panels (b) and (c) map the transitions between these three regimes as functions of the initial conditions. In panel (b), X_0 and Y_0 are varied with $Z_0 = 0.9$ held constant. In panel (c), Y_0 and Z_0 are varied with $X_0 = 0.5$. In both cases, the diagrams reveal a fragmented basin structure, where even small variations in initial conditions can shift the system from one attractor to another. This reinforces the notion that the model possesses a highly sensitive dependence on initial conditions, typical of systems exhibiting coexisting attractors and fractal-like basin boundaries [25,41].

The Basin Entropy analysis [34] provides a rigorous quantification of the basin fragmentation described above. For the chaotic-chaotic coexistence scenario ($\sigma = 30$, $r = 43$, Fig. 7), computing S_b over a high-resolution grid of 1001×1037 initial conditions yields $S_b(\epsilon) > 0$ for all tested box sizes ($\epsilon = 2, \dots, 50$ grid cells), confirming that the two basins remain intertwined across the full range of explored resolutions. At intermediate resolution ($\epsilon = 10$), $S_b = 0.35$, representing 51% of the theoretical maximum $\ln 2 \approx 0.693$, with 62% of local neighborhoods containing initial conditions converging to both competing chaotic attractors. At coarser scales ($\epsilon = 50$), S_b reaches 93.5% of $\ln 2$, showing that the basin boundaries permeate the entire initial-condition space.

By contrast, for the fixed-point/chaotic coexistence at $r = 10$ (Fig. 8), the Basin Entropy is negligible: $S_b(\epsilon = 10) = 0.001$, representing only 0.13% of $\ln 3$, with less than 0.3% of boxes containing mixed initial conditions. This two-order-of-magnitude difference ($S_b^{r=43}/S_b^{r=10} \approx 240$) quantitatively confirms that the basin fragmentation in the $r = 43$ scenario is consistent with a fractally intertwined, riddled-like structure, whereas the $r = 10$ case exhibits compact basins with smooth boundaries, two fundamentally distinct topologies of multistability within the same dynamical system [42,43].

The coexistence of multiple chaotic and non-chaotic attractors for fixed parameter values highlights the richness of the system's phase space and its relevance in the study of complex dynamical systems. This type of multistability is especially important in active matter models, where fluctuations or perturbations can lead to drastically different long-term behaviors depending solely on the system's starting point.

6. Conclusion

In this work, we have conducted a detailed analysis of the nonlinear dynamics exhibited by a minimal model for driven active matter, derived from hydrodynamic equations and extending the classical Lorenz system. The results support interpreting the present equations as a dissipative active extension of the Lorenz framework, not as a simple reparametrization of the passive convective problem. By combining Lyapunov exponents, isospike diagrams, complexity measures, and time-averaged values of the dynamical variables, we characterized both the global and local organization of the control parameter space. At the local scale, we reported for the first time the emergence of *quintuplet points* in this class of models, singular structures where five distinct periodic regimes meet at a single location in parameter space. These points act as organizing centers for spike-adding and spike-doubling bifurcations, revealing a form of discrete order in the topology of the control space. The recurrence rules governing the spikes and periods at the quintuplet points have been established and verified across various regions of the bi-parameter plane. Building on these local

structures, we demonstrated the existence of large regions of multistability where chaotic, periodic, and fixed-point attractors coexist under identical control parameters. We showed that the fine fragmentation of basins of attraction is directly related to final-state sensitivity in the presence of coexisting attractors, a phenomenon particularly relevant in low-dimensional active systems. Moreover, we demonstrated that the use of time-averaged observables, such as $\langle X \rangle$, serves as an effective indicator for classifying coexisting attractors and mapping their basins. Crucially, we quantified these fragmented structures using Basin Entropy analysis [34], revealing a two-order-of-magnitude contrast ($S_b^{r=43}/S_b^{r=10} \approx 240$) between strongly intertwined basins ($S_b/\ln N_A \approx 51\%$ for the chaotic-chaotic case) and compact basins with smooth boundaries ($S_b/\ln N_A \approx 0.1\%$ for the fixed-point/chaotic case). The approach presented here not only reinforces the value of traditional dynamical indicators but also introduces effective visual and quantitative strategies for analyzing transitions and multistable structures in active matter systems. These findings open new directions for the study of chaos, topological organization, and the control of nonlinear systems far from equilibrium.

CRedit authorship contribution statement

L. M. Pérez: Writing – original draft, Supervision, Conceptualization; **J. A. Vélez:** Writing – original draft, Software, Data curation; **J. F. Cueva:** Investigation, Data curation; **W. G. Marín:** Visualization, Data curation; **M. J. Suazo:** Methodology, Investigation; **P. Díaz:** Methodology, Investigation; **T. González-Vega:** Methodology, Formal analysis; **J. Bragard:** Writing – review & editing, Software; **D. Laroze:** Writing – review & editing, Project administration, Conceptualization.

Declaration of competing interest

The authors declare that there is not a conflict of interest.

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Data availability

Data will be made available on request.

Appendix A. Recurrence rules and quintuplet point analysis

This appendix provides the formal derivation of the recurrence rules governing quintuplet points (QPs), whose existence in the present model was established in Section 4. These rules, first identified in periodically driven chemical oscillators [37] and subsequently confirmed in several other dissipative systems [24,35,38], determine the spike counts and periods of all daughter phases that emerge at a QP.

Let us consider two adjacent periodic phases in the isospike diagram with spike counts n_1 and n_2 (where $n_1 < n_2$), separated by a smooth boundary. As control parameters are varied along this boundary, the system reaches a QP where the boundary trifurcates, giving rise to three new daughter phases at the next hierarchical level. The spike-count sequences and periods of these emerging phases satisfy the following recurrence equations:

$$n_{\text{add}} = n_1 + n_2, \quad (\text{A.1})$$

$$n_{\text{d},1} = 2n_1, \quad (\text{A.2})$$

$$n_{\text{d},2} = 2n_2, \quad (\text{A.3})$$

$$T_{\text{doubled}} \approx 2T_{\text{parent}}, \quad (\text{A.4})$$

where Eq. (A.1) represents the spike-adding rule (an analogue of the Farey median in number theory) which forms parabolic phases, and Eqs. (A.2) and (A.3) describes the spike-doubling rule forming cuspidal phases. As a result of these rules, the parabolic phase always exhibits an odd number of spikes ($n_1 + n_2$ is odd, as consecutive periodic phases typically have different parities), while the cuspidal phases always exhibit even spike counts ($2n_1$ and $2n_2$). When applied recursively, the parabolic arc n_{add} is flanked by the cuspidal phases $2n_1$ and $2n_2$, and two new QPs form at its endpoints with parents $(2n_1, n_{\text{add}})$ and $(n_{\text{add}}, 2n_2)$, producing a hierarchical cascade at the next level the form is $\{4n_1, 2n_1 + n_{\text{add}}, 2n_{\text{add}}, n_{\text{add}} + 2n_2, 4n_2\}$. In fact, in all three QPs identified in this work, the parent phases have consecutive spike counts $n_2 = n_1 + 1$. For this case, the cascade admits a closed-form solution. Denoting by $s_j^{(k)}$ the spike count of the j th phase at level k , one obtains by induction

$$s_j^{(k)} = (2^k - j)n_1 + jn_2 = 2^k n_1 + j, \quad j = 0, 1, \dots, 2^k, \quad (\text{A.5})$$

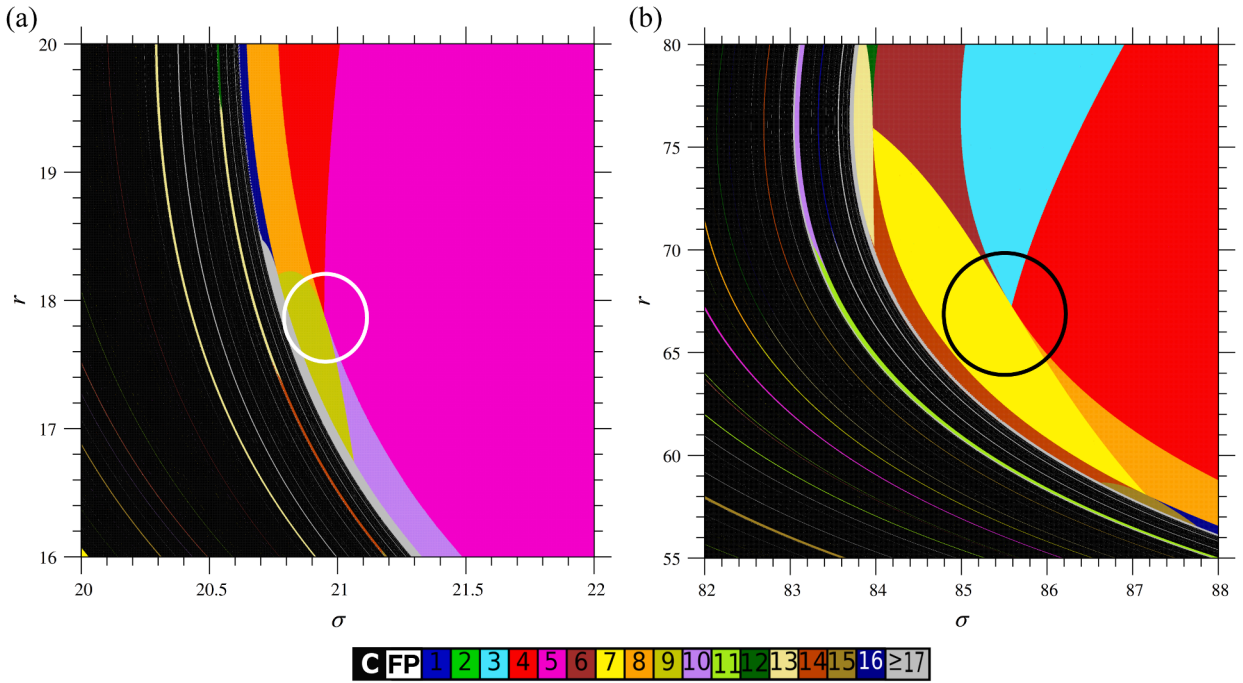


Fig. A.1. Isospike diagrams of the X component in two distinct regions of the parameter space, confirming the universality of the recurrence rules. (a) $\sigma \in [20, 22]$, $r \in [16, 20]$: the QP (white circle) is organized by parent phases $(n_1, n_2) = (4, 5)$, giving rise to adding (9) and doubling (8, 10) daughter phases. (b) $\sigma \in [82, 88]$, $r \in [55, 80]$: the QP (black circle) is organized by parent phases $(n_1, n_2) = (3, 4)$, producing adding (7) and doubling (6, 8) daughter phases. Despite different parent spike counts and control parameter ranges, both panels display the same self-similar structure of parabolic adding arcs and cuspidal doubling phases predicted by Eqs. (A.1)–(A.3).

so that the $2^k + 1$ phases at level k are the consecutive integers $\{2^k n_1, 2^k n_1 + 1, \dots, 2^k n_2 - 1, 2^k n_2\}$. Each phase is generated from the previous level by a recursive rule of Stern–Brocot type [44–46]:

$$s_j^{(k)} = \begin{cases} 2 s_{j/2}^{(k-1)}, & j \text{ even (doubling, cuspidal)}, \\ s_{(j-1)/2}^{(k-1)} + s_{(j+1)/2}^{(k-1)}, & j \text{ odd (adding, parabolic)}, \end{cases} \quad (\text{A.6})$$

with initial conditions $s_0^{(0)} = n_1$ and $s_1^{(0)} = n_2$. The parity constraint follows directly from Eq. (A.5): since $2^k n_1$ is always even for $k \geq 1$, the phases with even index j have even spike counts (cuspidal) and those with odd j have odd spike counts (parabolic), consistent with the observation that all parabolic arcs carry an odd number of spikes. At each successive level, the number of QPs doubles (2^{k-1} new QPs at level k) and the spike counts grow as 2^k , producing a self-similar tree that terminates when the periodic windows become narrower than the chaotic threshold.

To confirm that the QPs and their recurrence rules are a general property of the bifurcation topology of our system rather than an isolated phenomenon, we perform high-resolution parameter sweeps in two additional regions of the (σ, r) plane, as it is shown in Fig. A.1. In particular, this figure displays the isospike diagrams obtained from two different regions of the general (σ, r) plane. In panel (a) in the specific region of $\sigma \in [20, 22]$ and $r \in [16, 20]$, the isospike diagram reveals a QP (white circle) at the boundary between two parent phases with $n_1 = 4$ spikes (red) and $n_2 = 5$ spikes (magenta). At this point, the boundary trifurcates and three new daughter phases emerge: a parabolic adding phase with $n_{\text{add}} = 4 + 5 = 9$ spikes (olive), flanked by two cuspidal doubling phases with $2 \times 4 = 8$ spikes (orange) and $2 \times 5 = 10$ spikes (lavender), in exact agreement with Eqs. (A.1)–(A.3). Besides, in panel (b) in the specific region of $\sigma \in [82, 88]$ and $r \in [55, 80]$, a different QP (black circle) is identified at the boundary between parent phases with $n_1 = 3$ spikes (cyan) and $n_2 = 4$ spikes (red). Applying the same recurrence rules yields a parabolic phase with $n_{\text{add}} = 3 + 4 = 7$ spikes (yellow) and cuspidal phases with $2 \times 3 = 6$ spikes (brown) and $2 \times 4 = 8$ spikes (orange), all of which are clearly resolved in the diagram. Therefore, in both cases the rules are followed. These results, together with the (7, 8) QP analyzed in Section 4, establish that the adding-doubling cascade and the QP structure are robust topological features of the model. The same recurrence rules of Eqs. (A.1)–(A.5) govern the hierarchical organization of periodic windows regardless of the specific parent spike counts or the location in the parameter space.

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